DOCUMENT RESUME

ED 378 045 SE 055 604

AUTHOR	Buerk, Dorothy, Ed.
TITLE	Empowering Students by Promoting Active Learning in Mathematics: Teachers Speak to Teachers.
INSTITUTION	National Council of Teachers of Mathematics, Inc., Reston, Va.
REPORT NO	ISBN-0-87353-415-8
PUB DATE	94
NOTE	54p.
AVAILABLE FROM	National Council of Teachers of Mathematics, Inc., 1906 Association Dr., Reston, VA 22091-1593.
PUB TYPE	Guides - Classroom Use - Teaching Guides (For Teacher) (052) -- Collected Works - General (020) -- Books (010)
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Beliefs; College Faculty; *Concept Formation; Content Area Writing; *Cooperative Learning; Higher Education; Intermediate Grades; *Mathematics Instruction; Middle Schools; Secondary Education; Secondary School Teachers; Student Attitudes; Student Journals; *Student Participation; Students; *Thinking Skills; Writing Assignments
IDENTIFIERS	*NCTM Curriculum and Evaluation Standards; Reform Efforts; *Student Empowerment

ABSTRACT

This book is a collection of essays by full-time classroom teachers who describe their efforts to implement the vision of the National Council of Teachers of Mathematics Standards documents and to empower their students in mathematics. Chapter 1, "Students' Conceptions of Mathematics and the Challenge of the Standards" (Dorothy Buerk), gives an overview of how students often view mathematics as rote memorization and offers ideas for reforming this view based on the Standards. Chapter 2, "'Math Is Like a Used Car': Metaphors Reveal Attitudes toward Mathematics" (Helen Gibson), details a process that helps high school students think and write about their view of mathematics and themselves as learners. Chapter 3, "Cooperative Learning in the Secondary School Mathematics Classroom" (David Bock), explains how to organize students into cooperative learning groups to examine and verbalize mathematical ideas. Chapter 4, "Listening to Students through Writing" (Marcia Standera), and Chapter 5, "Writing and Cognition: Implications for Mathematics Instruction" (Patricia Ehrich), explain how writing activities in middle and high school classes can encourage students to think about math concepts, problems, and solutions. Examples of students' notes and journal entries are included in both chapters. Chapter 6, "Conflicting Goals in the Mathematics Classroom" (Ann Oaks), reviews attempts to help college students break out of rote memorization by using math exercises that require reasoning, inquiry, and exploration. Chapter 7, "Broadening Students' Conceptions of Mathematics: Theory into Action" (Dorothy Buerk), concludes the book with teacher guidelines for creating active student involvement in the math classroom. (MKR)

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Empowering Students

by Promoting Active Learning in Mathematics

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National Council of Teachers of Mathematics

Empowering Students by Promoting Active Learning in Mathematics

Empowering Students by Promoting Active Learning in Mathematics

Teachers Speak to Teachers

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Library of Congress Calaloging-in-Publication Data:

Empowering students by promoting active learning in mathematics : teachers speak to teachers / edited by Dorothy Buerk ; contributing teachers, David Bock ... [et al.].
p. cm.
Includes bibliographical references.
ISBN 0-87353-415-8
1. Mathematics—Study and teaching. I. Buerk, Dorothy.
II. Bock, David.
QA11.E66 1994
510'.71'2—dc20 94-39425
CIP

This work was supported by a grant from the Exxon Education Foundation for the project "Reclaiming Intuition in Mathematics," Dorothy Buerk, project director.

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Printed in the United States of America

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PREFACE

The enthusiasm and spirit of a group of dedicated teachers is contagious. The authors of this volume are such a group. We met on numerous occasions to define the direction of this collection of essays, to discuss drafts of the essays, and to emphasize points to be reinforced in its introductory and concluding chapters. Our discussions were lively, enlightening, and empowering. For each of us, sharing the spirit, enthusiasm, mathematical knowledge, depth of insight, and depth of caring of this group has been a highlight of developing this manuscript. This group spirit is reflected in the teacher-to-teacher focus of these essays.

As we talked about our own experiences as students of mathematics, we described how we had become comfortable and confident enough to play with mathematical concepts and ideas, to understand them for ourselves, to value them, and finally to own them. We examined how we had come to accept the theories we studied—a process that involved working and playing with them. We had tested theories by looking for exceptions and posing questions related to the theories. As we gained confidence in theories, we had looked for connections to other areas of mathematics. The theories we had studied were no longer someone else's theories; they had become our own. We agreed that we wanted our students to experience mathematics in these same ways; that goal led each of us to the teaching strategies and teaching philosophies that we describe here.

We agreed that many of our students need to broaden their conceptions of what mathematics is and how it can be learned in order to become the active, thoughtful, reflective learners that we are looking for in today's classrooms. Many of the views of mathematics and classroom strategies that had worked well for these students in other settings would no longer be appropriate in these new environments. We discussed ways of changing students' conceptions and realized that each of us had made progress toward this goal in a unique way. At this point, common themes began to emerge from our discussion that we wanted to share with other teachers.

Our next task was to capture our best thinking on paper. As we wrote about our experience, we became aware of the process of writing: gathering our ideas, trying to organize them in some fashion, writing them down, revising (sometimes in major ways, sometimes in minor ways), and then editing the draft. We found this process difficult and often got discouraged, but the feedback we got from each other helped us to revise again until we were satisfied with what we had written. We became aware also that this writing process was the same process we used in mathematics and that we wanted our students to learn to use it in mathematics as well. Our empathy for our students and their courage deepened.

Essential to our project was the inspiration provided by the hundreds of mathematics educators who contributed to drafting the *Curriculum and Evaluation Standards for School Mathematics*. We are also indebted to the Exxon Education Foundation, whose generous funding of our project, "Reclaiming Intuition in Mathematics," supported our work. We benefited from the advice of Hadley Smith, of the Writing Program at Ithaca College, our writing consultant and editorial coordinator. In closing, we thank Janet Kalinowski, who read many drafts, made numerous suggestions for revision, and offered her valuable insights. Alan Sherman, Avery Solomon, and Stephen I. Brown offered helpful guidance as well. We are particularly indebted to Barbara Warland, June Hannah, and Penny Rice, whose technical expertise was invaluable in completing this manuscript.

David Bock
Dorothy Buerk
Patricia Ehrich
Helen Gibson
Ann Oaks
Marcia Standera

1

INTRODUCTION

Students' Conceptions of Mathematics and the Challenge of the *Standards*

Dorothy Buerk

> Doing math can result in a precise answer or an estimate but *it is not really a thinking process.* Rather it is a process of identifying, comparing, and doing a problem in relationship to that identification and comparison.
>
> —Lee, *a college freshman*

> Unlike English class, math was not a place for ideas in process. You could not say or share something you were thinking about. You could only share with the class completed, perfected thoughts, and I simply had no such thoughts concerning math.
>
> —Jackie, *a college graduate, in reflection*

Both Lee and Jackie believe that their own thoughts are not appropriate in mathematics. To survive in a mathematics classroom, Lee has developed a mechanical strategy to approach mathematics and Jackie has become silent. Students like Lee and Jackie challenge me to find ways to help them change their view of what mathematics is and to help them become confident enough to share their mathematical thoughts and ideas with me and with their classmates.

The *Curriculum and Evaluation Standards for School Mathematics* (hereafter called the *Standards*), issued in March 1989 by the National Council of Teachers of Mathematics (NCTM), makes clear that the views of mathematics held by Lee and Jackie impede their mathematics learning. Classroom teachers have long recognized this impediment. Because of their views of mathematics, students reduce the entire subject to a series of algorithms to be memorized and reproduced, and they resist any encouragement to listen to their own mathematical ideas.

In this collection, five classroom teachers—Helen Gibson, David Bock, Marcia Standera, Patricia Ehrich, and Ann Oaks—who are successful in empowering their students, in implementing the recommendations in the *Standards*, in improving their own classroom practices, and in changing both their students' conceptions of mathematics and their classroom behavior share their approaches, their strategies, and their experience. This material is not designed to be a set of lessons that you can import directly into your curriculum. Rather, in the spirit of the *Standards*, it presents

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All names, except Jackie's, used in these essays are pseudonyms we have given to the students whom we quote or discuss.

the experience of successful teachers that you can blend with your own experience. You can then decide which strategies and activities presented here are appropriate for you and for your classroom and how you might modify them to meet the needs of your students. Before these teachers speak, I want to (1) look more closely at students' views of mathematics and their implications for classroom practice, (2) examine the *Standards* and the resistance that is part of any attempt to change classroom practice, and (3) summarize the ideas for implementing the vision of the *Standards* that this volume has to offer you as a classroom teacher.

SOME STUDENTS' VOICES

Lee offers further insight into his conception of mathematics:

> The math process is one in which all attention is focused on a narrow subject. My mind is not allowed to create, wander, or think about doing the problem. My mind says, "Compare this problem to others that are like it and base your answer on the way you found your answer to that other problem."

Lee has learned, though he may not have been taught, to do mathematics by rote and has rejected mathematics *because* it does not allow for his own thinking. To identify and to compare may sound to us like "thinking skills," but for Lee they are part of a mechanical process. He believes that learning mathematics is learning to reproduce exactly someone else's methods and results. He has many skills but no confidence that he will know when to use them. In his essay, "Cooperative Learning in the Secondary School Mathematics Classroom," David Bock presents his Ten Myths of Mathematics, a system of beliefs similar to the one Lee has adopted. (For more on Lee's experience, see "Writing in Mathematics: A Vehicle for Development and Empowerment" [Buerk 1990].)

Jackie completed calculus in high school but considered herself a "math phobic." In other disciplines she valued the process through which ideas are developed but believed that in mathematics that process must be private and that her evolving or partially formed ideas were inappropriate in a mathematics classroom. (For Jackie's reflection and more on her experience, see "Getting beneath the Mask, Moving out of Silence" [Buerk and Szablewski 1993].) Both Lee and Jackie describe views of mathematics as a discipline that excludes their own thinking and that requires rote learning. Similar views are held by many students. Some function successfully using rote techniques and may even become frustrated when asked to explain their methods. Others who lack confidence in their ability to perform mathematically find this rote view particularly debilitating. They no longer try anything on their own; they reject their intuition and common sense and both avoid and fear the study of mathematics. In her essay, "'Math Is a Like a Used Car': Metaphors Reveal Attitudes toward Mathematics," Helen Gibson presents examples of students who believe that they have no control over their mathematics learning.

Once students adopt the rote view of mathematics, it becomes the basis of their approach to our classes and undermines our attempts to lead them beyond algorithms to conceptual understanding. The rote view is self-perpetuating and addicting. Ann Oaks, in her essay, "Conflicting Goals in the Mathematics Classroom," describes how students and teachers can be at cross-purposes because of their differing views of, and expectations about, mathematics learning.

I ask my students to write about their strategies for learning mathematics. Many share what I call survival strategies for a mathematics classroom, including the following:

- "I'll just do my assignments and go to class without actually participating or trying to solve the problems." *(This student's in-class plan is to be silent and watch.)*
- "I figured that I'd attempt to do the problem, not be able to, and then *wait* for the answer to be explained to me so I'd have some idea of what was going on." *(This student believes she cannot solve the problem and so will not try for long. She also believes that her teacher will not wait very long before he or she discusses the answer anyway.)*
- "I write down the first thing that comes to my mind, just to get it over with." *(This student does not want to think about the problem at all!)*
- "It is better not to try than to try and be wrong." *(The passive role allows this student to keep some of his self-esteem.)*

I had seen the passive behavior; all mathematics teachers have. Until I began to collect my students'

explanations, I had not been aware that so many of them had *chosen* that behavior. Some made it clear to me that they had learned from their past experiences that the passive role was the safest. Some, like the last student quoted above, have been intimidated or felt "stupid" in mathematics classes in the past and hope to avoid those feelings by their passive behavior. Others believe that they are in fact *expected* to be passive in their mathematics classes. In response to this inertia, Marcia Standera has developed several writing techniques that help students become more actively involved in her classes. She describes these strategies in her essay, "Listening to Students through Writing."

In addition to being passive, Lee and Jackie believe that doing mathematics in school does not involve their own thinking. Many other students accept rote processing in mathematics without concern. Recently, in a class discussion on exponents, a freshman told me that exponents were added when multiplying factors with the same base. I asked him why. He said, "That's the rule." I asked him why the rule said that. "It just does," he replied. "It's the rule I was taught." "But why?" I persisted. He looked at me very seriously and asked, "You mean there's a reason?" He was sincerely surprised to hear that there might be some rationale for this mechanical manipulation. In her essay, "Writing and Cognition: Implications for Mathematics Instruction," Patricia Ehrich describes compelling questions that help her middle school students think on paper and develop their own rationales for mathematical situations.

Another student who believes that she does not need to think in mathematics classes writes the following:

> For me math is most like a machine. You plug in numbers to formulas, equations, objects, and out comes an answer. Math does not require analytical thinking, nor do machines. You just turn them on, plug them in, and you have a definite, precise, factual answer. The motions (of machines) and answers (in mathematics) are mechanical with no critical or analytical thinking involved.

Some students with this belief can manipulate numbers and symbols very well. Others cannot. Many, however, are debilitated by the belief that mathematics offers no opportunities for their own critical or analytical thoughts, and they end up passive, reluctant learners. Students who believe that "only completed, perfected thoughts" can be uttered in a mathematics classroom will become passive and may lose confidence in their ability to learn mathematics if no "perfected thoughts" are forthcoming. Even when these students do solve problems on their own, they do not value their skill because they see mathematics as a domain where they must be given formulas and procedures to follow precisely. They, like the students Ann Oaks describes in her essay, believe that problems they can solve by reasoning or intuition must not be mathematical problems.

In order to become true students of mathematics, many pupils need to see connections among their thoughts, the mathematical concepts they are studying, and their everyday surroundings. They need to recognize that reasons exist for the rules, methods, and algorithms they study and to understand that these reasons are accessible to them. They need to start to own and value their ideas, their questions, and their thoughts. They need to transform their present view that mathematics is a finished product that results from reproducing exactly someone else's thinking into a view of mathematics as a dynamic, creative process. They need to listen to their own thoughts, questions, and insights as aids in making sense of algorithms, procedures, theorems, applications, and proofs. They need to create their own ideas, to try them out, and to test them. They need to express their own unfinished ideas while those ideas are still developing. They need to see that David Bock's Ten Myths of Mathematics are truly myths. Each of the essays that follow gives examples of students actively involved in this transformation process.

Students who do not listen to their own thoughts but continue trying to manipulate symbols mechanically without understanding may leave the study of mathematics or may become dependent on their teachers. They may give to us a power that we neither want nor value. Recently this happened to me quite unexpectedly. I often ask my students to explain to a peer in writing a mathematical idea or concept, and so I asked the following question on an examination:

> A student is solving a pair of equations by the addition method, and when she adds the equations together, she obtains $0 = -3$. The student has done the work correctly. Please explain to the student precisely why this happened and what it says about the equations.

One student responded in this way:

> The lines will be parallel to each other. The reason why is because it is understood. You are our teacher and you told us that if we see something like (0 = −3), the lines are parallel. After all, if you can't trust your teacher, well then school is a waste of time.

This student had listened to me when I showed him that a contradictory conclusion like 0 = −3 occurred when the equations represented parallel lines. He found my request for a reason inappropriate. I had told him what was true. I was the authority from his point of view, and there was no reason for him to explain. "The reason why is because it is understood." This student memorizes well, follows procedures well, and accepts my explanations. My question was, from his point of view, not only inappropriate but also meaningless. This particular student saw all his teachers as final authorities in their disciplines and did not use his own thoughts in any course he took. More often, however, students become dependent when they do not believe that they can use the same strategies in mathematics classes that they use in their other courses and in other areas of their lives. They are reluctant to rely on their thinking and their ideas in process, and so they become dependent on their teachers for answers. They may, in fact, become angry when asked to explain their reasoning, since that request contradicts their view of how mathematics is learned. Despite the strength of the self-perpetuating, rote view of mathematics, some students like Lee and Jackie become empowered and excited about mathematics when they experience it in the way described in the *Standards*. Each of the five essays in this collection describes ways to offer this experience to students.

THE STANDARDS

The *Standards* acknowledges that mathematics learning has historically been equated with the rote memorization of rules. Many teachers, education administrators, parents, and students like those I have described still connect success in mathematics with the unthinking recitation of facts, rules, formulas, and algorithms. In fact, those in other disciplines often contrast their emphasis on interpretation, uncertainty, analysis, and reflective thinking with what they believe is an emphasis on absolutes and certainty in mathematics. Many, both in and out of mathematics, believe that David Bock's Ten Myths of Mathematics are true. Helen Gibson gives us a means to ascertain the beliefs of our students as a first step toward modifying their views.

Many teachers have long believed that students learn best by doing and that they must be given the opportunity through mathematical situations to explore, to conjecture, to think, to reason, to justify, to verify, to describe, and to explain using their own words. Helen Gibson, David Bock, Marcia Standera, Patricia Ehrich, and Ann Oaks all present examples of mathematical situations in which students are given these kinds of opportunities. Support from the *Standards* has been invigorating and rewarding, and the *Standards* has become a rallying point for teachers determined to change how students learn mathematics. In fact, the authors of this collection of essays came together originally to help other teachers begin to implement NCTM's recommendations.

The misconceptions about mathematics learning are well entrenched in our society; in the expectations of parents, students, and administrators; and in our own classroom experience. Teachers know that this image of mathematics is incorrect, inappropriate, and often debilitating and that it must be changed. I believe that this change will be very difficult to accomplish and that innovators will meet with resistance more often than they expect.

Sometimes, for example, pressures created by expanded curriculum content, demands for higher standardized test scores, larger classes, and reduced support staff may tempt teachers to take the easy route and to do their students' thinking for them. In frustration, teachers may encourage students' passive behavior even while knowing it is not in anyone's best interests.

We must also be wary of the pitfalls that resulted from past attempts to reform and improve mathematics education. Problem solving, although a well-intentioned approach, has often become just a series of rote instructions that students must execute in a specified order rather than a process for using intuitive, reflective, and analytic skills.

IMPLEMENTING THE VISION OF THE STANDARDS

In the essays that follow, five full-time classroom teachers describe their efforts to implement the vision of the *Standards* and empower their students. The publication of the *Standards* and the

funding from the Exxon Education Foundation brought us together to discuss ways in which we could share our experience with other mathematics teachers. As we met and examined what we were doing in our classrooms, we realized that in different ways each of us was addressing the five general goals for all students that were set in the *Standards* (NCTM 1989, p. 5):

1. That they learn to value mathematics
2. That they become confident in their abilities to do mathematics
3. That they become mathematical problem solvers
4. That they learn to communicate mathematically
5. That they learn to reason mathematically

In "'Math Is Like a Used Car': Metaphors Reveal Attitudes toward Mathematics," Helen Gibson describes how having her high school students develop their own figurative language for mathematics offers them a means of reflecting on and clarifying their thinking about mathematics as a discipline. The metaphors serve as a vehicle for communication between student and teacher that reveals the students' level of confidence in their mathematical abilities and the extent to which they value the discipline.

David Bock analyzes in "Cooperative Learning in the Secondary School Mathematics Classroom" the Ten Myths of Mathematics that prevent his high school students from thinking as mathematicians. He explains how to create productive cooperative learning groups in which students verbalize mathematical ideas, focus on their abilities as mathematicians, appreciate the power and beauty of mathematics, and move beyond the myths.

In "Listening to Students through Writing," Marcia Standera describes how she uses writing with her high school classes as part of her regular classroom routine. She finds that this process aids students in thinking through problems; makes them more accountable for their problem solving; and opens new levels of communication from student to teacher, from teacher to student, and among her students.

Patricia Ehrich, in "Writing and Cognition: Implications for Mathematics Instruction," discusses writing in mathematics, focusing on the evidence of cognitive processing that results from written activities. She shows that writing helps her students think and reflect, affirm their own understanding of mathematical concepts, explore a range of approaches to problems, and experience an "aha"—a spontaneous discovery about mathematics.

In "Conflicting Goals in the Mathematics Classroom," Ann Oaks reviews her work with college students who conceive of mathematics as a set of procedures that they must learn by rote and who do not understand that it represents both a systematic body of knowledge and a method of reasoning. She discusses her attempts to help them move away from their dependence on instructors as the source of all mathematical knowledge and begin to rely more on their own thinking as a source of mathematical ideas and thereby experience ownership of those ideas.

The five essays are written to stand alone, each in the teacher's voice. Taken together, they provide guidance, strategies, and support for teachers who are implementing the vision of the *Standards*, improving their classroom practice, actively engaging their students, broadening their students' conceptions of mathematics, and changing their classroom behavior to empower students.

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2

"Math Is Like a Used Car": Metaphors Reveal Attitudes toward Mathematics

Helen Gibson

"For me, math is most like an unreliable car." Imagine mathematics as a used car, a book, a toolbox, or a bullfight. These are among the ways in which my students have described mathematics. Comparing mathematics to specific objects enables students to focus on their views of mathematics and of themselves as learners. Collecting these mathematics "metaphors" (*metaphor* is being used in a broad sense in this essay) is a unique and quick way for me to learn about my students' conceptions of mathematics. The results are always interesting and sometimes surprising, and they can have lasting effects in the classroom. This essay gives an overview of the metaphor process, including actual student responses and how-to guides for collecting mathematics metaphors from your own students.

When students exhibit passive attitudes and behaviors in mathematics class, teachers have cause for concern. The prevailing attitude among many students seems to be "Give me the formula and some numbers, and leave me alone." This "fast food" approach to mathematics could be attributed to society, video games, music videos, or possibly a state curriculum sequence consisting of fragmented units and objective, plug-into-the-formula test questions. For whatever reasons, students seem less apt to be active participants in the learning process and more apt to sit back and wait for the formula or answer.

Even a beginning teacher can easily perceive this attitude, as I learned during the 1987–88 school year when I was privileged to work with a talented student-teacher who was assigned to work with my two precalculus classes. Within the first few days of his teaching experience, he had observed that we were dealing with two entirely distinct classes: the "Thinkers" and the "Robots," as he called them. The Thinkers were a unique group, an impatient, questioning crew but not top students by any means. They asked reckless questions and weren't afraid to make mistakes. (They made many mistakes.) Sometimes they would make conjectures and argue among themselves until they were convinced of the validity of statements. The Robots, who came in the very next period, were a passive bunch whose immediate goal was to get that night's homework assignment. They would politely listen to the lesson, ask if they were to be tested on the material (if I said yes, they'd quickly jot down something), and start the homework. For the most part, these were the honor students, the accelerated juniors who would make up the next year's Advanced Placement calculus class. They lived by the motto "Give Me the Formula," and my fear was that they would also die by it. A deeper fear was that most of my classes fit this Robot description.

In a conversation with Dorothy Buerk, I expressed my concerns about the prevailing attitudes that the student-teacher and I had been observing. She encouraged me to learn more about my students' conceptions of mathematics through a writing

exercise that she had developed and used with her college students for several semesters. In this exercise, students would describe mathematics using a metaphor; they would complete the sentence "For me, math is most like a(n) _____" and then explain their choice of metaphor in one or two paragraphs. Working together, we adapted the process for students in grades 9–12 and scheduled a classroom exercise.

Before the exercise began, we primed each class by telling the students that we were interested in their responses to a set of directives concerning their views of mathematics. (We never used the word *metaphor* in our discussions with the students.) The directives in figure 2.1 were given first orally, then by writing them one at a time on the chalkboard.

> List all the words or phrases you would use to describe mathematics to a friend.
>
> (*Wait 5–7 minutes.*)
>
> Imagine yourself doing or using mathematics either in or out of school; list all the feelings that come to mind.
>
> (*Wait 5–7 minutes.*)
>
> List all the objects (nouns, things) that you think math is like.
>
> (*Wait 5–7 minutes.*)
>
> Read over your three lists carefully (*allow a lengthy pause here*). From the third list, choose the word that best describes what math is most like for you. Finish this sentence: "For me, math is most like a(n)_____." Explain your choice fully in a paragraph.
>
> (*Allow 15–20 minutes.*)

Fig. 2.1

When the papers were collected, students had the option of signing their names or submitting their papers unsigned; only two chose to submit anonymously.

STUDENTS' ATTITUDES AND CONCEPTIONS

The first set of metaphors revealed how students perceived not only mathematics as a field of study but also themselves as learners. Some of these revelations were surprising to me. Many viewed mathematics as something beyond their control:

> For me, math is like falling down a giant waterfall. Going down is fun and easy, but once you hit the bottom, you are thrown about by the force of the water. It's a challenge to see if you can "beat" the water and survive or whether you will just give up and let the water get the best of you. Eventually if you stick it out you move on to calm waters, and everything is fine—until the next fall.

> For me math is a tree. Math is seemingly calm and easy. However, at difficult times it is like a hurricane. It wrenches up the roots of your learning like the roots of a tree. When math becomes easy, the storm subsides, the roots settle back in, and your tree of learning grows.

> Math is most like an earthquake. If an earthquake was to hit, eve.. just a tremor, it could knock down and ruin a lot of things. Just like in math, if you make one error in a problem, even a small one, it can ruin or tear down all of your work.

Other students wrote about intense feelings of anger, despair, and frustration; they believed that they had little control over their own learning:

> For me, math is like a giant jigsaw puzzle, with all the pieces the same color.

> For me, math is like an endless, dark tunnel. So much is to be learned, but it is so difficult to understand.

> For me, math is most like a fog because you cannot see or understand where it came from, but you just have to keep moving forward. You can only move forward slowly, otherwise you'll get lost.

> For me, math is like an unreliable car because sometimes it works and sometimes it doesn't. You are constantly working on it. It usually breaks down, and it makes me mad.

> For me, math is a never ending list of rules that doesn't really mean anything. Math seems to be the most indecisive subject there is. In the beginning we were taught basic math, like, you can't subtract four from one. That seemed logical to me. Then they told us that you could do that, but it equals a negative number. A negative number is less than zero. Zero is nothing. How can anything be less than nothing?

During my first reading of the metaphors, I focused only on the negative aspects (e.g., frustration, despair, and powerlessness). As I reread the metaphors and the accompanying lists of adjectives and nouns, however, I found many positive responses:

For me, math is like climbing a huge mountain. It is terribly difficult going up, yet when you get there, you know you've accomplished something.

For me, solving math problems is like surfing on a never ending wave. You can ride the wave as long as you solve all the problems in your way.

To me, math is like a used car that you can get for a good price; sometimes it runs smoothly, but on certain days things go wrong. It's frustrating, like a car can be, when it won't go right. You have to sweat, yell and curse, and sometimes pay a price to get the car going, but once it does go, everything's great. With math, things don't always work out right. I don't know how many times I've screamed and pulled my hair out trying to "fix" a math problem, but when I finally figure it out, I feel fantastic, like I've accomplished something. Sometimes you break down, in a car or during a math problem, but if you work with it, you'll get to where you're going!

These students wrote about the excitement of working on a challenging problem and the feeling of exhilaration on completing it. The sense of accomplishment was very satisfying and, in many instances, enough of an impetus to keep working when the next obstacle appeared.

Many of these responses included both the positive and the negative extremes. It was obvious that students felt frustration and despair when they thought they had no control over their successes and failures. It was equally obvious that they experienced joy and satisfaction when they successfully completed a challenging problem.

I decided to schedule a small block of discussion time into my next day's plans. Without exception, each class began with students' eagerly asking about my reaction to their papers ("Did you read mine yet?" "What did you think about ...?" "Now you know how I feel about ..."). When I returned their metaphors with comments or questions written in the margins, they didn't hesitate to explain what they had written. We talked about such words as *frustration* and *powerlessness*. Most students viewed frustration as part of, and even somewhat necessary for, learning. For them, the sense of powerlessness occurred only when they experienced heavy doses of frustration with no relief in between them. We spent much of each period discussing metaphors and how best to learn and teach mathematics.

A closer analysis of the metaphors and the lists of words generated from the Thinkers and Robots revealed some interesting comparisons. Well over 60 percent of the Thinkers said that they got "excited" or felt "exhilarated" when doing mathematics. Only 30 percent of the Robots used those or similar words. Words like *happy*, *satisfied*, and *challenging* appeared far more often in the Thinkers' lists. The Robots tended to use words like *uninteresting* and *difficult* and seemed to focus on the need to get the right answer. The Thinkers—the reckless questioners who would probably not continue in mathematics—enjoyed doing mathematics. The Robots from the accelerated track, who were already scheduled for AP calculus, did mathematics to get an answer. This system of doing and learning mathematics had worked for them for years. By conventional standards, they were successful. I had no doubt that they would continue to be successful in higher-level mathematics courses, but they would lack that sense of enthusiasm and excitement that sparked the Thinkers. It was clear to me that if the Robots were to become more like their Thinker counterparts, their attitudes toward mathematics needed serious attention. They had to get beyond simply finding the answer. They needed the confidence and reckless abandon of the Thinker who wrote about the used car. She often experienced frustrations but knew that she could be successful. She had ownership of her learning, and that made all the difference.

BENEFITS FOR THE TEACHER

Collecting metaphors had a profound effect on me. The implications were obvious: students needed to become more actively involved in the learning process to become Thinkers. I couldn't "fix" the negatives and eliminate the sense of powerlessness by myself. I could, however, create a positive environment in which students could risk small doses of frustration. I could furnish meaningful lessons and activities that involved all students and relied more on cooperative learning. I could help students develop coping strategies and problem-solving skills that would enable them to have more control over their learning. And, being able to match a metaphor to a face, I could fine-tune my support for each student. I had my work cut out for me.

The Thinkers and Robots of 1988 have long since graduated from high school. I continue, however, to collect metaphors from all my students early in

September each year. The benefits gained by collecting and discussing the metaphors are great in comparison with the brief time spent on this exercise. By learning about my students' conceptions of mathematics, I am better able to direct my energies to their needs. The students start the year with a more accurate picture of themselves and, knowing my concerns for them as learners, are more willing to depart from the traditional homework-lecture-homework approach that they've experienced before. They participate more in cooperative learning activities that get them involved immediately and enable them to draw on their own resources and those of their classmates.

Like most humans, high school students find it difficult to change old habits and patterns. Although Thinkers and Robots alike resist abandoning the motto "Give Me the Formula," they are adapting to a different pattern of learning as we work *together* to become learners and teachers of mathematics. Formulas, homework, and tests are not handed to students by teachers in order to get the "right" answer; they are tools to empower students to learn. As students experience success in overcoming their frustrations with the mysteries of mathematics, they become more willing to change old habits of passivity and adopt new patterns of problem solving, knowing that they *can* have control over their own learning.

I urge you to collect metaphors from your own students. Before you start, however, you may find helpful the "Suggestions for Gathering Mathematics Metaphors" (appendix A) and "Protocol for Gathering Mathematics Metaphors" (appendix B). Both are found at the end of this article. They have been revised as we have gained more experience in the metaphor process. A final word of encouragement (and warning): Be prepared for an eye-opening experience that will change the way you and your students view each other. Good luck!

Appendix A

SUGGESTIONS FOR GATHERING MATHEMATICS METAPHORS

Dorothy Buerk and Helen Gibson

If you want to gather mathematics metaphors from your students, you have several decisions to make.

1. *Why do you want your students' metaphors for mathematics?*

You may want to use the metaphors to set the stage for a discussion of the diverse views of mathematics held by those in your classroom. In this instance, you should return the papers so students can talk about what they wrote. Some students like to read theirs, others like to talk about theirs, others will not refer to their metaphors but will enter the discussion, and others will remain silent. If your students are working in cooperative learning groups, you may want this discussion to take place in the small groups that the students will work in for a time. Sharing in this way could help the groups to get acquainted.

You may want to use the metaphors as a gauge of your students' changing conceptions of mathematics. In this instance, you should gather them early and late in the term, semester, or academic year. You will probably not want to return the first set when they are written. You may, however, want to return both papers to the students late in the year and ask them to write about the similarities and differences that they see in their metaphors. Students see their changing views and are quite articulate in describing them.

You may use the metaphors to help you respond to each student individually. If this is your goal, you still need to decide if you want to return the papers to your students.

2. *Decide whether to comment on the metaphors that you receive.*

This writing represents your students' spontaneous, honest responses to the questions asked, but it was not written to be evaluated. You may, however, want to write supportive, understanding comments or questions on them, or you may prefer to return them as you received them.

3. *Who will read the metaphors the students write?*

Students need to know that what they write is read in confidence. They need to know who will read what they write *before* they write. They need to know that their permission is required before their writing is shared with others.

4. *Decide how to set the stage for the activity.*

Writing comparisons for mathematics will be a new experience for most of your students. You should tell them why it is important to you that they share with you what they really think. After you have set the stage, you might introduce the activity in the following way:

> Today I want you to think about mathematics and write about it. I will guide your thinking. I want each of you to write what you *really think*, rather than what you think I might believe or what you think I may want to hear. I ask each of you to remain silent during the writing activity so that everyone has a chance to think and write. When we have finished, you can share with your friends if you wish. I would like you to hand in what you have written at the end of the period so that I can read it. (If appropriate, add the following.) I will return your writing to you at our next class meeting.

Note: Avoid the word *metaphor* with your classes. For some, the word has no meaning; for others, the meaning is technical. The use of the word might interfere with your students' ability to think freely on paper.

Appendix B

PROTOCOL FOR GATHERING MATHEMATICS METAPHORS

Dorothy Buerk and Helen Gibson

In the classroom, each student writes his or her own responses to each of the following oral directions. We also write the directions on the chalkboard after they are given verbally.

1. Pretend that you have to describe mathematics to someone. List all the words or phrases you can think of that you could use. (*Wait 5–7 minutes.*)
2. Imagine yourself doing or using mathematics either in or out of school. What does doing or using mathematics feel like? List all the words or phrases you would use to describe what doing or using mathematics feels like. (*Wait 5–7 minutes.*)
3. Think about the things that mathematics is like. List all the things or objects that you think mathematics is like. List each object you choose on a separate line on your paper. (*Wait for a few minutes.*)
4. Let me suggest some categories that you might want to think about to extend the list of things that you think mathematics is most like.
   - If math were weather, what kind of weather would it be? (*Wait after each suggestion.*)
   - If math were a food, what food would it be?
   - If math were a food, how would you eat it?
   - If math were something you can do in your free time, what free-time activity would it be?
   - If math were a way to travel, what means of travel would it be?
   - If math were a color, what color would it be?
   - If math were a way to communicate, what way would it be?
   - If math were an animal, what animal would it be?
   - If math were a building, what kind of building would it be?
   - If math were a plant, what kind of plant would it be?
   - If math were music, what kind of music would it be?
5. Now go back over your list of things that mathematics is like and comment briefly on why each thing is like math for you. (*Wait.*)
6. Read over your list of words that describe mathematics. (*Wait.*)
7. Read over your list of the feelings you have when doing mathematics. (*Wait.*)
8. Now go down your list of things that mathematics is like, saying, "Math is (*state each word*)." (*Wait.*)
9. Choose the object on your list that best fits mathematics. Think about the ways that mathematics and this object are alike for you.
10. Write a paragraph beginning "For me, math is most like a(n)_____." In that paragraph discuss the ways in which your object and mathematics are similar. (In classes for whom writing paragraphs may be difficult, we ask students to list all the ways in which their object of choice and mathematics are alike.)

3

Cooperative Learning in the Secondary School Mathematics Classroom

David Bock

Too often students get the wrong messages in their mathematics classes. They learn what I call the Ten Myths of Mathematics:

Myth 1: Mathematics is true or false, right or wrong.

Myth 2: Mathematics involves mostly calculations and equations.

Myth 3: Every question has an answer.

Myth 4: Every question has only one correct answer.

Myth 5: There is only one way (or at least a preferred way) to find that answer.

Myth 6: This solution probably involves a formula, a step-by-step procedure, or a trick.

Myth 7: Someone has to tell you what to do.

Myth 8: Real mathematics is formal: definitions, theorems, proofs, formulas, and so on.

Myth 9: Mathematics is a static body of knowledge compiled long ago.

Myth 10: Mathematics is only marginally related to the real world.

We all recognize, of course, that these myths are false; even worse, they are harmful. They create a perception that obscures the beauty, the joy, and the power of mathematics. They place the teacher at the intellectual center of the classroom, playing the role of dictator: think my thoughts, use my methods, get my answers, try to be me. They make the student a passive recipient of meaningless rules for playing someone else's game.

There is another view of mathematics. Mathematics is an approach, a way of exploring and describing the world of nature and the world of the imagination. Mathematics is alive, flexible, and inherently interesting. Rather than obscure, most of secondary school mathematics is fundamentally obvious. Yes, occasionally something counterintuitive is revealed, but the mere fact that we find such rare insights so remarkable attests to the commonsensical accessibility of most mathematics concepts. In this view of mathematics, the role of the teacher is not to cover ideas but to help students uncover them. The role of the students is to be active learners, which requires using intuition, common sense, and insight; explaining; listening; asking questions; clarifying; and taking risks. Active learners develop personal understandings of the concepts, appreciation for the power and beauty of mathematics, and confidence in their ability to learn.

Pie in the sky? Sure! It doesn't happen that way day in and day out in any classroom. However, I find that cooperative learning techniques provide effective ways to supplant the myths with this more rewarding and empowering view of mathematics.

TAKING THE FIRST STEPS

Throughout my teaching career, I had occasionally tried to undermine the ten myths using open-ended problems, projects, and laboratory types of investigations. Although I used these methods infrequently, I saw their potential benefits, which left me eager to try more. Then I attended a one-day cooperative learning workshop after which I read Neil Davidson's (1990) *Cooperative Learning in Mathematics*. Inspired by this introduction, I read more about cooperative learning, and then I began to use those techniques in my high school mathematics classes. My students are more enthusiastic about this approach; they really enjoy their mathematics classes and are developing positive attitudes about mathematics and themselves.

To get started using cooperative learning in your classroom, there are but two fundamental requirements: some courage and a little knowledge. Courage is required to overcome pedagogical inertia. Although it was easier and safer to dust off a lesson I had used a dozen times before than to create a new approach, I had to be willing to experiment and to take some risks. It was like starting over as a first-year teacher—but a wiser first-year teacher. Did I have bad days? Sure. But traditional pedagogy is certainly no guarantee against bad days. And I have found that I don't have to be creative and clever every day. I haven't abandoned everything I've ever done but instead have expanded my repertoire. I still use many traditional lessons, but the more often I think of an effective cooperative activity, the more my students adjust to the approach and the more ideas I have for future classes. Classrooms with students in straight rows of chairs and teacher-dominated instruction now seem artificial and lifeless. The transformation I've witnessed in my students is worth all the effort and risks.

LEARNING ABOUT COOPERATIVE LEARNING

Initially, some basic knowledge of cooperative learning ideas and techniques is indispensable. I attended workshops and did some reading; a few excellent resources are listed in the bibliography. I learned about the following ways of organizing student interaction, which are explained in Kagan (1992): roundtables, think-pair-share activities, numbered heads together, jigsaws, relays, and others. I sought information specifically supporting the use of cooperative learning in a mathematics classroom (Davidson 1990).

Although books and workshops were helpful, my biggest problem was finding sources of ideas, lesson plans, problems, and other activities designed to illuminate the topics I teach. At times I was able to think of clever and innovative ways to turn what would have been a teacher-centered lecture or demonstration into a cooperative exercise, but more often (especially at first) I couldn't. Few published resources (see Serra [1989], London [1989], and the *Investigations* series [Posamentier and Sheridan 1982a, 1982b, 1983]) are available. I have found that one productive and enjoyable way to address this shortage is to band together with other mathematics teachers to share ideas. We formed a "cooperative teaching" group we call CLAM, Cooperative Learning Alliance for Mathematics. CLAM meets one evening a month, bringing together about twenty teachers from fifteen different schools in a dozen nearby districts for a show-and-tell session. We exchange classroom materials, explain lessons we've used, discuss what worked well and what did not, and brainstorm ways of addressing upcoming topics. As we model the cooperative learning process in our own group, we get many good ideas and build a library of locally generated materials.

REDESIGNING THE CLASSROOM

To restructure my classroom for cooperative learning, I have to change both the physical arrangement of the desks and the dynamics of classroom interactions. I want to vary activities quickly and easily from a teacher-led discussion to work in pairs or groups of four and to avoid having any students seated with their backs to the chalkboard or the projector screen. I cluster the desks as shown in figure 3.1, angled toward the front center of the room, with two students facing forward and two facing each other. I use this very flexible arrangement for every class unless a test or a quiz is scheduled.

However, change requires more than just moving the furniture; it requires redefining the roles of student and teacher. This redefinition involves forming the groups, fostering effective group interactions, and ensuring that students understand the process

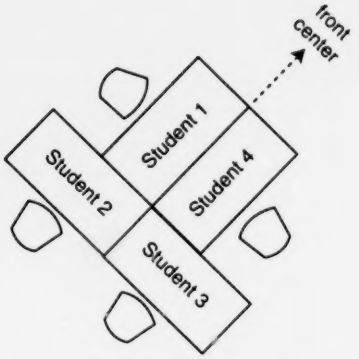

Fig. 3.1

and the objectives. Simply having students sit together during traditional teacher-centered instruction is not cooperative learning.

CREATING PRODUCTIVE GROUPS

Among the several ways to form learner groups are assignment by the teacher, students' choice, and random selection (Kagan 1992; Davidson 1990; Johnson 1986). No matter how the groups are created, students must learn to work with one another productively. At first there may be confusion and even resistance; students are not accustomed to working together, especially in mathematics. However, studies show that the main reason students come to school is to see their friends. Rather than spend the day fighting this desire to socialize, enlist it to get students involved in subject-related activities.

I start by taking time to explain my objectives, discuss the anticipated benefits of cooperative learning, allow students to express misgivings, and encourage them to give the method a fair trial. I explain the desired group dynamics through a discussion of the following "group guidelines" displayed on a poster in my room:

1. Work TOGETHER—cooperate with other group members.
2. LISTEN carefully to everyone's questions and ideas.
3. Be sure EVERYONE PARTICIPATES and NO ONE DOMINATES.
4. Achieve a GROUP SOLUTION—be sure EVERYONE UNDERSTANDS.

Next, before turning my attention to the coursework, I engage the students in group-building activities designed to develop cohesiveness, effective interaction, and good communication skills. I have students interview each other in pairs and introduce their partners to the other group members. Then, to move in the direction of mathematics, I have them solve a few jigsaw problems from Tim Erickson's (1989) *Get It Together*. Each student gets one of four necessary pieces of information; only by discussing their clues with one another can the group find a solution. These exercises are very effective in involving everyone and suppressing the "designated thinker" phenomenon. Finally, except in very rare instances where a group doesn't gel, I leave these groups together for several weeks to allow productive dynamics to emerge and grow.

GETTING ON WITH THE COURSE

I teach the same mathematics as always, but whenever possible, I find ways to involve students in group investigations, discussions, and other activities intended to make them active participants in creating their own understanding of the topics. Two examples that illustrate how my approach has changed follow.

I used to introduce the slope-intercept form of the equation of a line through demonstration and lecture. I would stand at the chalkboard and graph a line by making a table of values and plotting some points. I would take care to point out how the coefficients in the algebraic expression were evidenced in the resulting graph. Through subsequent examples, I would ask the students leading questions as I carefully produced more graphs. Eventually I would have the students work individually on some practice examples, then I would answer their questions and assign the homework. This approach was successful: the students learned to graph lines using information about the slope and the y-intercept. However, this approach also reinforced several of the myths. Students perceived that doing these graphs required knowledge of a trick that I had to reveal to them as they first watched passively and then simply mimicked my approach.

Now I get the students involved at all stages of the lesson, which I have adapted from the ideas brought to one of our CLAM meetings by Patricia Ehrich. I give the groups the equations of four lines. Each

student in the group is assigned one of the lines and makes a table of coordinates of points in preparation for a graphing roundtable. Taking turns, they graph all four lines on a single sheet of graph paper passed around the group. They can then compare graphs and record their observations. First I give them a set of equations that produce parallel lines with varying y-intercepts. We repeat the process with sets of lines for which the y-intercept is the same but the slope varies. After analyzing the four sets of equations shown in appendix A, the students use their observations to try to predict the graphs from the given equations and finally reverse the process by trying to guess the equations after inspecting the graphs. As the period nears an end, we engage in a student-centered full-class discussion, summarizing and clarifying the important concepts they have discovered.

The students learn the same mathematics as before concerning the slope-intercept form of the equation of a line, but much more happens. The students are *actively* involved in the class. They practice graphing lines, helping each other when necessary. Together they make observations, formulate conjectures, and engage in productive discussions. They laugh and argue with their classmates, questioning others' methods and ideas and clarifying their own. Most important, they discover and apply new insights *themselves*.

Another example is my new approach to teaching right-triangle trigonometry. I used to introduce this topic with the definitions of sine, cosine, and tangent; a discussion of opposite and adjacent legs; attention to the table of values in the textbook; and numerous problems. I succeeded, but students had to wrestle simultaneously with new vocabulary, new concepts, and a scary table. This was another lengthy teacher-dominated presentation combining important topics with a large dose of the myths: definitions, formulas, and ancient knowledge revealed by an authority.

I now begin by assigning each pair of students an acute angle. Using a centimeter ruler and protractor, they draw a set of five similar right triangles containing their assigned angle. Each pair measures the base, altitude, and hypotenuse of each of their triangles. Then they use a calculator to find the ratios of these sides. They observe, for instance, that all the ratios of altitude to base are close, and they approximate the actual constant value by averaging their data. They then use this result to help each other solve a problem I've given them that involves their angle. While I circulate, offering help and advice where needed, I record the ratios approximated by each pair of students. The class has created a table for most angles from ten to eighty degrees in five-degree intervals, which I quickly duplicate and distribute. Then as a whole class, we do a couple of standard guy-wire or length-of-shadow problems, and I send the students away with the set of homework exercises given in appendix B. The next day they start class by discussing their answers in their groups, and I clarify any important issues for the entire class. Typically they express fears that their results could be inaccurate and complain that their data omit most angles. That's the perfect moment to show them the trigonometric table and introduce the terminology *sine*, *cosine*, and *tangent*.

Once again, students learn the same mathematics as before, but as active learners. They enjoy working together and helping one another. They draw, measure, and calculate, generating their own data. Students appreciate how and why trigonometry works because they generate their own understanding of the concepts and experience success in applying them. They get quite excited when checking the table they created against the real thing and are impressed with the accuracy of their approximations. They see that the vocabulary of formal trigonometry describes familiar and meaningful phenomena. Math makes sense.

CHANGING CLASSES, STUDENTS, AND TEACHERS

When you start using cooperative learning, some things change immediately; others evolve as you and your students gain experience in this new approach. As time passes, you cease to be the center of attention. You spend less time reteaching, clarifying, and reviewing poorly understood material. Your job becomes preparing activities and guiding investigations. You run from group to group facilitating discussions, clarifying sticky points, and arbitrating deadlocks. When you finally get around to summarizing and perhaps formalizing the concepts of the day, you enjoy a more attentive and productive atmosphere because your students, having already thought about the ideas, have a base of experience to which they can relate your explanations and an intellectual investment in the issues.

Your students become active learners; they much prefer *doing* mathematics to *watching* you do it. Rather than write what you say in a notebook to be tossed in a locker and never seen again, they think and talk with one another about the mathematical concepts. Mathematics becomes an acceptable topic of conversation both in and out of class. Your students develop better mathematics vocabularies and improved clarity of expression. They learn to listen to other people's suggestions and questions. They become teachers themselves as they explain their ideas to others in their group. And their base of experience broadens as they hear alternative explanations and see a wide variety of analytical approaches.

As you rely more on cooperative learning, your students' confidence in their ability to do mathematics grows. They see that intuition, conjecture, and common sense are of value; that everyone makes mistakes; and that everyone has good ideas and insights. They realize that it's not just the teacher who understands, and they become more willing to tackle tougher problems and topics. As individuals they may succumb to a thirty-second frustration factor, giving up if a productive avenue doesn't quickly appear, but rarely does a whole group simultaneously admit defeat. Instead, there seems to be an intellectual critical mass in which someone always has an idea. And, empowered by their successful experiences in class, rarely do they lament, "I thought I understood, but I couldn't do the homework."

Cooperative learning lessons can address the mathematics you need to teach while directly combatting the Ten Myths of Mathematics. At first you may fear that the class moves too slowly, that you won't finish the course. Stick with it! As I explained these approaches to one of my colleagues, he went to the chalkboard and drew a lake, identifying two points on opposite shores. "I start here in September," he said, indicating one of the points, "and then lead and herd my class around the end of this lake in a feverish rush to arrive over here by June. You're telling me that instead I should be teaching them to swim." That's exactly right. As students build deeper understanding and take greater responsibility for their own learning, the pace accelerates. You must start the process by overcoming the reluctance to get your own feet wet!

Am I enthusiastic about cooperative learning? Obviously. But beware: it's not a panacea. It won't end absenteeism, cure drug abuse, or turn your weakest student into a National Merit Scholar. It can, however, establish a framework for a greater variety of classroom activities, involve students in the learning process, and change their views of themselves and of mathematics. They actually *enjoy* mathematics; I feel it daily in my classes. In the year-end evaluations, students complain of not enough group work, frequently mention the benefits I discussed earlier, and use descriptions like "best class" and "fun." And finally, instead of believing that mathematics is imposed and formal, they come to see it as a process involving curiosity, investigation, conjecture, intuition, refinement, and justification. One student summarized this enthusiam best: "I never thought math could be a class where I'd get to use my imagination."

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Appendix A: GRAPHING LINES

I use the activity sheet on the following page for the investigation of the slope-intercept form of the equation of a line. The cooperative learning activity is described below:

1. The numbers 1–4 in each set of equations correspond to the seats of the students in the group, thus automatically assigning an equation to each student.
2. The students must already know how to create a table of coordinates from an equation and how to graph a line based on the table.
3. The teacher must prepare in advance, and distribute to each group with the accompanying activity sheet, four sheets of graph paper clearly labeled "Set A," "Set B," and so on.
4. During the discussion that kicks off this lesson, the teacher must point out that each student will be graphing one line on each sheet of graph paper as it is passed around the group.
5. Since there are four sets of equations and four sheets of graph paper, each student should be continuously busy creating a table, graphing a line, or helping someone else.
6. After all the sets are graphed, the students should begin examining the graphs one set at a time, recording their observations and conjectures.
7. The students then begin applying their insights by working together on the last two sections.
8. When most groups have finished the bulk of the exercises, the stage is set for a productive class-wide discussion.

Graphing Lines—What does the equation reveal?

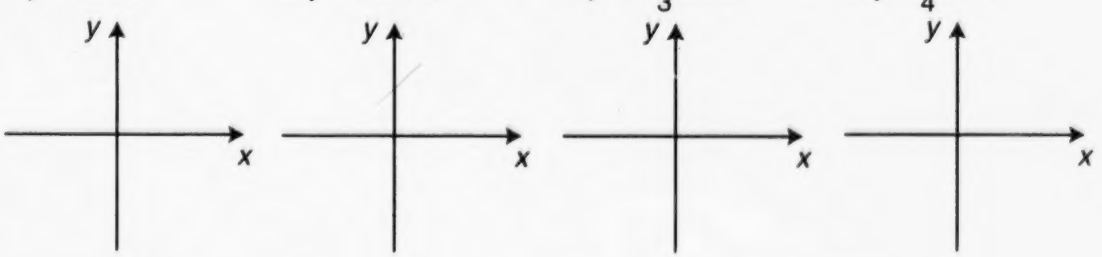
1. For each set of equations, each person in the group should make a chart to find the values of x and y for his or her assigned equation, and then graph and label the line on the appropriate axes. The graph for each set will then show four lines. What does the equation reveal about the graph?

Set A	**Observations:**	**Set B**	**Observations:**
1. $y = 2x + 1$		**1.** $y = 1x + 2$	
2. $y = 2x + 6$		**2.** $y = 2x + 2$	
3. $y = 2x - 2$		**3.** $y = 3x + 2$	
4. $y = 2x - 5$		**4.** $y = 4x + 2$	

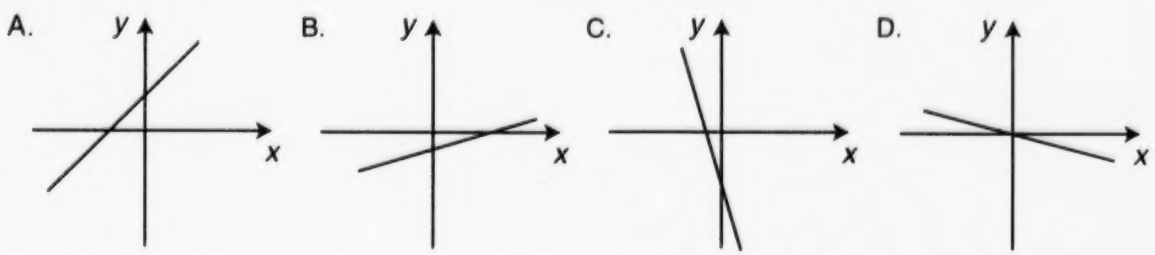
Set C	**Observations:**	**Set D**	**Observations:**
1. $y = 2x - 1$		**1.** $y = \frac{1}{2}x + 3$	
2. $y = 1x - 1$		**2.** $y = 3x + 3$	
3. $y = \frac{1}{2}x - 1$		**3.** $y = -2x + 3$	
4. $y = \frac{1}{4}x - 1$		**4.** $y = -4x + 3$	

2. Use the observations you made to help you predict the graph of each line. Record your prediction here, then make a chart and draw the graph. How close was your prediction to the actual graph?

A. $y = 3x - 2$ B. $y = -1x + 5$ C. $y = -\frac{1}{3}x - 8$ D. $y = \frac{3}{4}x + 1$



3. **Jeopardy:** Examine each graph; what might its equation be?



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Appendix B: RIGHT-TRIANGLE TRIGONOMETRY

I use the activity sheets on the next four pages to implement the right-triangle-trigonometry lesson. Some additional advice may be helpful:

1. Students are paired as usual: Student 1 with Student 2, and Student 3 with Student 4.
2. Each pair receives a protractor, a centimeter ruler, and a calculator.
3. Each pair receives a set of instructions (sheet 1) specifying its assigned angle. Before class, write a different angle on each sheet: 15°, 20°, 25°, and so on. Since the smaller angles and their complements are hard to work with, the extreme angles can be omitted if the class has too few students to assign all angles in five-degree intervals. Although you may not have enough students to do a complete set of angles, be sure to assign all the angles that the students will need for the homework assignment.
4. Each pair also receives a sheet to draw on (sheet 2) that has the right angle and two sides calibrated in centimeters.
5. Note that instead of using the traditional trigonometric terms *opposite* and *adjacent legs*, I use the geometric terms *base* and *altitude*. The students know what these words mean, so they can develop an understanding of the new concepts *before* encountering potentially confusing new vocabulary.
6. As the pairs of students begin to complete their approximations of the ratios, record the results in a table like that on sheet 3. The completed table can be copied for each student in the class. A goal of this lesson is to mimic a trigonometric table. Therefore, the calculations are carried out to three decimal places even though that level of accuracy is not warrented by the original data. Figure 3.2 shows the results of one class's calculations, most of which are reasonably accurate.
7. Be sure to work a couple of examples with the entire class so that everyone is comfortable with the use of the table, which is used in the set of problems (sheet 4) assigned for homework.

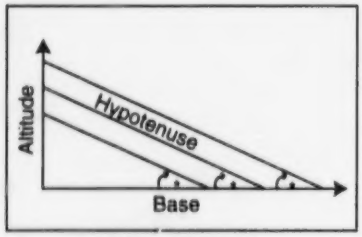
Angle	$\frac{\text{Altitude}}{\text{Base}}$	$\frac{\text{Altitude}}{\text{Hypotenuse}}$	$\frac{\text{Base}}{\text{Hypotenuse}}$
5°			
10°			
15°	0.266	0.256	0.962
20°	0.363	0.335	0.934
25°	0.73	0.429	0.906
30°			
35°	0.732	0.573	0.810
40°	0.841	0.646	0.768
45°			
50°	1.18	0.757	0.642
55°	1.45	0.830	0.566
60°	1.789	0.874	0.489
65°	2.106	0.907	0.433
70°	2.64	0.933	0.355
75°	3.68	0.960	0.263
80°			
85°			

Fig. 3.2

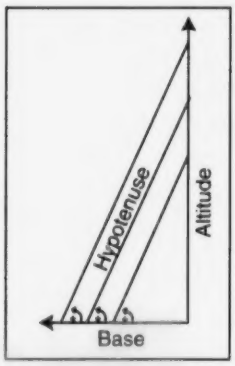
Measuring Right Triangles — Sheet 1

Your Angle = _____.*

1. Carefully draw a series of right triangles (at least five) having *your* angle of elevation. This will work best if you have your right-triangle-construction sheet turned as shown.
2. Measure the base, altitude, and hypotenuse of each triangle. Approximate these distances to the nearest tenth. Record these lengths in the table below.
3. Calculate and record the specified ratios for each triangle. Use your calculator. Round off to the nearest thousandth.



Angles less than 45°



Angles greater than 45°

Triangle	Lengths of Sides (to nearest tenth)			Ratios (to nearest thousandth)		
	Base	Altitude	Hypotenuse	$\frac{\text{Altitude}}{\text{Base}}$	$\frac{\text{Altitude}}{\text{Hypotenuse}}$	$\frac{\text{Base}}{\text{Hypotenuse}}$
1.						
2.						
3.						
4.						
5.						
Approximated actual value of each ratio:						

4. Although the three ratios will be different, the values in each column should be close. Why?

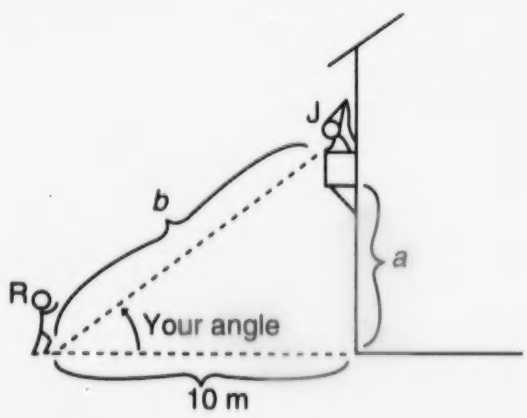
5. Average your results to approximate the actual value of each of the three ratios.

6. Now solve this problem:

   Romeo stands 10 meters from Juliet's house to serenade her as she stands on the balcony above. The angle of elevation from Romeo to Juliet is *your angle*.

   (*a*) How high off the ground is she?

   (*b*) How far apart are these lovers?



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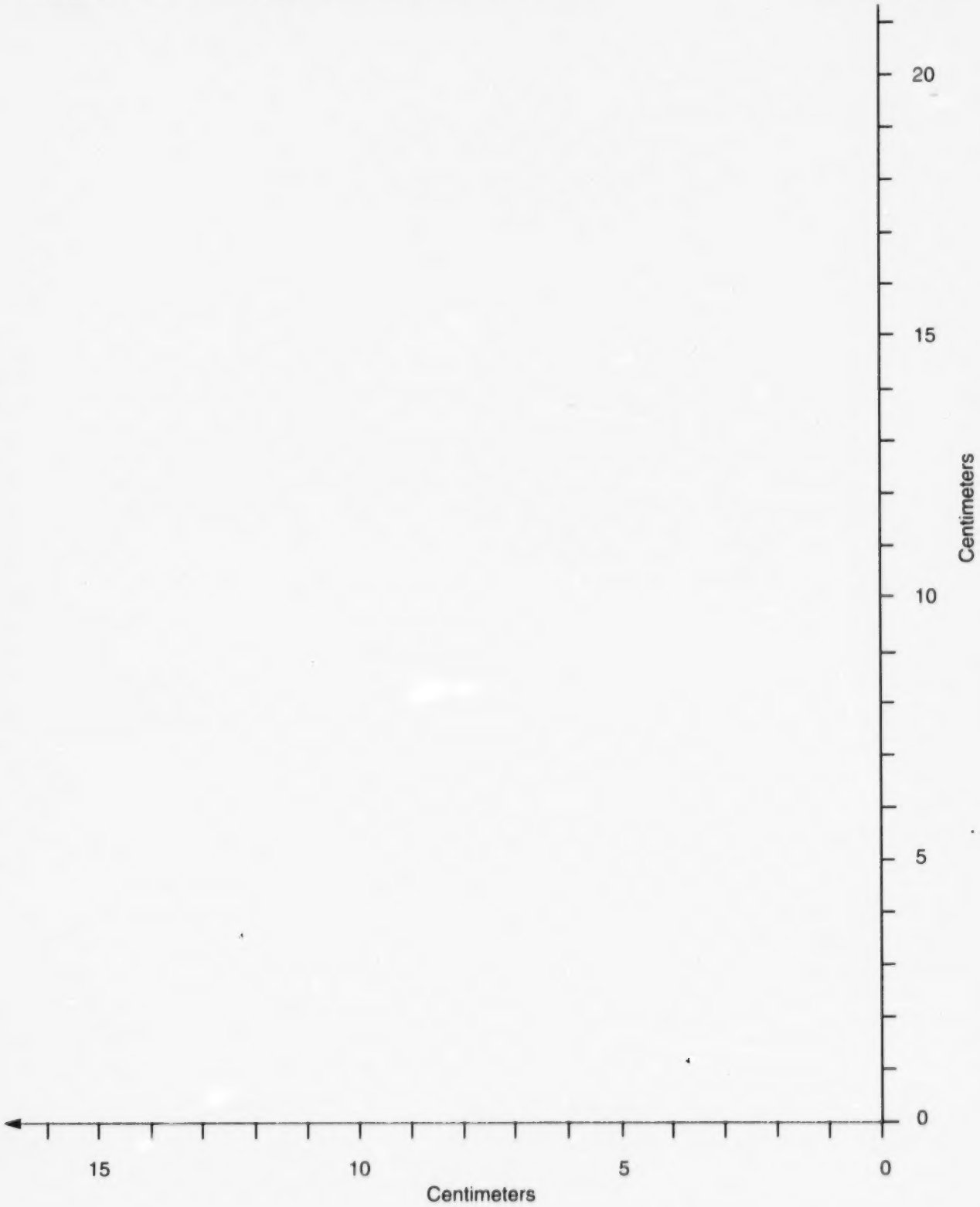
Right-Triangle Construction

Sheet 2



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Approximated Ratios

Sheet 8

Angle	$\frac{\text{Altitude}}{\text{Base}}$	$\frac{\text{Altitude}}{\text{Hypotenuse}}$	$\frac{\text{Base}}{\text{Hypotenuse}}$
5°			
10°			
15°			
20°			
25°			
30°			
35°			
40°			
45°			
50°			
55°			
60°			
65°			
70°			
75°			
80°			
85°			

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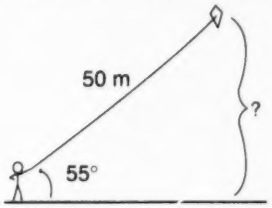
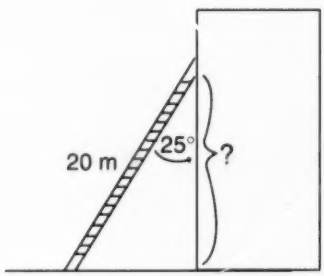
Homework:

Using the list of ratios you created in class, solve each of these problems. Give answers to the nearest tenth (or to the nearest angle).

1. When a child flying a kite has let out 50 meters of string, the string makes an angle of 55 degrees with the ground. How high is the kite?

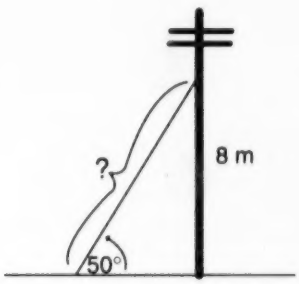


2. A ladder 20 meters long makes an angle of 25 degrees with the side of a building. How far up the wall does the ladder reach?



3. When the angle of elevation of the sun is 35 degrees, a flagpole casts a shadow 30 meters long. How tall is the flagpole?

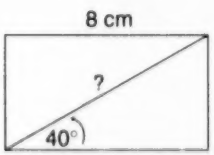
4. A guy wire must be attached to a telephone pole 8 meters above the ground and should make an angle of 50 degrees with the ground. How long must the wire be?



5. Find the base angle of an isosceles triangle with sides 20, 29, and 29 units.

6. An airplane flying at an altitude of 14 000 feet is sighted at an angle of elevation of 15 degrees. How far away is it? (Line-of-sight distance)

7. How long is the diagonal of a rectangle if it makes an angle of 40 degrees with a side 8 centimeters long?



8. An isosceles trapezoid has a base of 30 centimeters, an altitude of 7 centimeters, and a base angle of 75 degrees. Find the shorter base.

9. Find the area of a parallelogram with sides 8 meters and 10 meters and an angle of 60 degrees.

10. A regular 9-sided polygon is inscribed in a circle 20 centimeters in diameter. Find the area of the polygon.

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4

Listening to Students through Writing

Marcia Standera

Writing is a tool I use to expand my understanding of how my high school students feel about mathematics, how they solve problems, how they process information, and how they view themselves as learners. Writing helps me individualize instruction and enhances group learning. It strengthens the lines of communication with my students, makes them more accountable and responsible for assignments, increases their conception of mathematics as a process rather than just arriving at answers, and encourages creativity in problem solving. An underlying goal of all the writing techniques I use is to have my students present their ideas, problems, and solutions clearly and precisely. This skill is transferable to any discipline.

TECHNIQUES

Writing begins on the first day of class when my students fill out a standard school form. In addition to the usual information, I ask them to complete the following three open-ended sentences, which I write on the chalkboard:

1. The thing I like most about math is _____.
2. The thing I like least about math is _____.
3. If I could change one thing about math class, _____.

I collect these forms and immediately read them all. This exercise not only establishes the idea of writing about mathematics the first day but is a quick method for me to obtain some information about how each of my 120 students feels about mathematics. By responding verbally or in writing to my students during the first week of school, I establish an immediate line of communication that tells my students I care about them and value what they write.

I continue the writing process throughout the year by having my students write where they "got stuck" on assigned problems. Each day, I either collect assignments or circulate around the room and check them. To complete an assignment, students must do every problem or write one sentence or more of explanation for each place where they "got stuck." Many times, describing what they don't understand helps them think about the process of solving the problem, and they "unstick" themselves. Writing about where they "got stuck" also makes them more accountable for assignments. No longer can students hand in blank sheets of paper and say, "I don't understand." They have to think about what they don't understand and express it in written form.

After a few weeks of school, my students feel comfortable writing about where they had difficulty on a problem, and I emphasize in class that problems typically can be solved in multiple ways. At this point, I introduce another writing technique that I call "verification." Verification is a much

broader concept than "checking" and requires that students explain why they think their answers are correct. They can do a "check" or find an alternative method of solving a problem. Clarity, completeness, and presentation of solutions are emphasized as essential to successful verification. Some topics in mathematics lend themselves to creative verifications; others do not. I have students verify one problem for each assignment when appropriate, and I usually include one verification on each test. Computer-generated solutions, manipulatives, contradiction proofs, and graphs are just a few of the ways in which students verify answers. Verifying fosters creativity in problem solving.

Students keep their writing exercises, along with all other assignments, notes, quizzes, and tests, in a loose-leaf notebook. I collect the notebooks and grade them once a marking period according to neatness, organization, and completeness of assignments. The notebooks allow me to see the progress my students have made in writing and in understanding mathematical concepts. I write comments to all my students, answer some of their written questions, and comment on the strengths and weaknesses of their assignments. The notebook offers a means for one-to-one communication with my students and serves as a record of their accomplishments.

COMMUNICATION

Writing establishes an excellent line of communication between me and my students. It gives me some valuable information about all my students, not just the ones that easily share their thoughts and ideas in conversation. This communication begins on the first day of class when my students answer the three open-ended questions. From the information gathered in this brief exercise, I recognize misplaced students, make schedule changes, determine which students are "mathphobic" and need special care, identify the strong writers, find out which students don't like to do homework, identify work-study students for the mathematics department, and select students for my mathematics team. This writing exercise is also a test for me as a teacher. My students must have faith that I read their comments and that I am honestly interested in them. If writing is to continue as a form of communication in my classroom, then I must respond to their comments.

Often students have trouble talking to teachers, and writing gives them an alternative means for communicating their feelings to teachers. For example, Christi, a quiet third-year mathematics student, wrote the following answers to the three open-ended questions:

1. The thing I like most about math is it's pretty easy; it comes easy to me.
2. The thing I like least about math is all the equations I have to memorize.
3. The thing I would change about math class is the people in it.

The third statement bothered me. Why would a girl who is well liked by her peers not want to be with the students in a standard junior mathematics class? Was it because the level was too easy for her? In an effort to solve this puzzle, I talked to her previous mathematics teachers and found out that Christi wasn't an active participant in class, although she was an excellent mathematics student. After asking them why they thought that Christi wasn't placed in an honors mathematics class, their common response was, "I knew she was good, but I didn't think she was *that* good." It became clear to me that Christi resented not being in the honors program, and thus she did not feel motivated to participate in class. In a few days I talked to Christi, and even though she was very reluctant to discuss her third response, she was eager to accept my suggestion to transfer into honors mathematics, where she succeeded. Her short first-day writing exercise furnished the key to her problem in the mathematics classroom.

Sometimes a significant number of students in the same class give similar responses to one of the open-ended questions. This alerts me to a classwide perception that may affect the atmosphere in the class. For example, many of my general-mathematics students did not have a very good image of themselves as mathematicians. Typical was Josh's response:

1. The thing I like most about math is nothing.
2. The thing I like least about math is I'm in school.
3. If I could change one thing about math is I don't know.

From this response and his previous grades in mathematics, I could tell that Josh perceived himself to be a weak student, and he needed much positive reinforcement. From talking to Josh, I found out that his mother wasn't good in mathematics,

and he thought he took after her. As teachers, we must help students overcome these mindsets. After much hard work and encouragement, Josh successfully passed the course.

Sometimes classwide responses lead to a discussion that improves the atmosphere in the class. For example, my honors precalculus class wrote that they disliked being tested before they felt they knew the material. After a discussion about who was responsible for preparing the class for tests, the students admitted that they could work harder to prepare themselves and that the preparation wasn't all up to the teacher. This discussion made me more sensitive to students' test anxieties, and as a result I sometimes spend one extra day on the material before testing.

I learn a lot about my students through their writing. Even the limited information I receive from their first-day exercise has allowed me immediately to remove some simple barriers to learning. It has opened a line of communication between me and my students that is strengthened as we correspond through their notebooks during the year.

I also encourage my students to communicate with me on their homework assignments. Originally, I had my students write a sentence or two explaining where they "got stuck" on a problem to make them accountable for completing assignments. If they couldn't complete a problem, they weren't allowed to leave it blank and merely stop work but rather had to describe their difficulty. This technique accomplished much more than just making my students more responsible. I also learned about their personalities and how they solve problems.

At the beginning of class when I go around the room to check for completed assignments, I sometimes notice that several students have written sentences indicating that they had difficulty with the same problem. I might start the class with that problem and have the students read their sentences to the class. Many times their sentences will be similar, and the students understand that others experience similar confusion or difficulty. Also, some of my weaker mathematics students are exceptional writers, and they get to read a strong paragraph to the class describing their difficulties. The other students compliment them on how well they expressed their thoughts, thus giving them positive reinforcement. In addition to writing, I have had students draw cartoons to describe where they "got stuck" on a problem, and I even had a student tape a song he wrote called "Mathday Blues" in which he describes the pitfalls of problems involving series and sequences. All these students received full credit for their assignments, and I got to know them a little better. They were working with mathematics in their own terms and expressing their thoughts and feelings to me. This illustrated to them that mathematics is a process, not just an answer to a problem.

Writing about assignments also alerts me to problems that students are having outside the classroom. For instance, one of my students wrote a paragraph explaining that a family member had been in a car accident and thus his assignment for my class would be late. In instances such as this, students open up to me personally through writing, so I understand how their personal lives affect their classroom performance.

Students also communicate to me when they verify problems on assignments. When I introduce the technique of verification to my students, my goals are to make my students more confident in their answers and to get them to explore alternative methods of solution. I not only accomplish these goals but learn more about the way in which my students view the process of solving problems. Many of my students believe the myth that correct approaches to problems come only from the teacher. It takes time for students to get the courage to be creative. They are afraid of failure. However, when they experience success in discovering a unique way of verifying a problem, the risk of failure is worth it.

A good example of verification comes from my precalculus class. I asked the students to find the equation of a conic section from a locus-of-points description. Most students solved the problem by using analytic techniques. Verifications involved graphing the equation so that it satisfies the locus description, plotting the equation on the computer and then checking the locus, or plotting specific locus points and using them in the general conic equation. Some students wrote a paragraph describing how they roughly sketched the locus and through a process of elimination determined which type of conic section the equation represented. They told me that they couldn't prove that the equation was correct but that they knew it was a parabola, since their equation described a parabola. When I work with students on verifications such as these, we share moments of mathematical discovery that are truly exciting.

BENEFITS TO STUDENTS

The benefits of writing in mathematics class are not immediately obvious to students. At first they write because it is required, but later writing becomes a part of their learning processes. As the course progresses, my students come to appreciate the value of their writing experience in mathematics.

One benefit is that writing enhances group learning. For example, during a study hall I supervise, students sometimes call me over to ask me questions about an assignment with which they all had difficulty. They share their sentences on where they "got stuck" on a problem and in the process communicate their understanding of mathematics to one another. This adds to the effectiveness of their cooperative learning efforts.

Through writing verifications, students feel ownership of, and pride in, their creative solutions. For example, solving quadratics is a topic rich in different methods of solution. So far, my students have verified their answers by factoring, substituting, applying the quadratic formula, using the sum and product of the roots, deriving computer approximations, and estimating the x-intercepts of the parabola $y = ax^2 + bx + c$. On occasion, my students come up with original verification methods that we have not covered in class, some of which lead to solution techniques that can be generalized. My students now feel ownership of their learning, which leads to better retention of the material. They have become true problem solvers and thinkers, and they are also much better at applying mathematical concepts to real life.

Writing as part of their assignments has made my students more responsible and independent. Now, they are accountable for *all* problems, even those that they are at first unable to solve. Since they have to spend the time to write about where they had difficulty, they are likely to spend more time trying to solve the problem.

Students also benefit from having a notebook for written exercises and other assignments. It organizes their work and establishes an excellent reference source. They take pride in their notebooks and are amazed at all the work they have accomplished in a ten-week period. The organizational skill needed to maintain the notebook is transferable to problem solving in mathematics.

BENEFITS TO TEACHERS

When I began to use writing in my classroom, the most immediate benefit to me was that I got to know my students as quickly as possible without using too much class time. The writing exercise I use on the first day of class is quick but gives me real insight into the makeup of my classes. My students and I begin communicating immediately.

Writing also supplies me with a tool to overcome the frustrating time barrier imposed by the class period. It is almost impossible to answer all my students' questions and still teach a new lesson in a class period. This situation eventually frustrates students, and some stop asking questions, which is tragic. Writing allows the dialogue to continue outside of class and encourages the students' participation.

The students' writing also helps me monitor the effectiveness of my teaching. For example, students' written comments sometimes highlight concepts or facts that were not clear to them in class and that need to be taught using a different approach. I am now aware of students' frustrations with a class or their assignments. As a result, I have become a more flexible teacher.

Writing exercises have allowed me to increase my expectations for my students, and they have met the challenge. My students have impressed me with their creative solutions and their flexibility in problem solving. When taking a test, they no longer give up on a problem when they can't remember an algorithm, but they try to solve the problem by alternative means.

A final benefit of writing in the mathematics classroom is that it results in students' caring more about their work. When you show the students you care about their ideas and are interested in them as people, they produce for you.

CONCLUSION

As a teacher, the rewards I have received from incorporating writing into my classes far surpass the extra effort expended in using these techniques. Asking my students to fill out a form the first day of class, to write sentences describing where they "got stuck" on problems, and to verify their answers on assignments has opened up new lines of communication between my students and me. I have become

a better teacher and my students have become better learners.

I challenge you as a teacher to try some of these writing techniques in your own classroom. Create your own open-ended questions or try mine. Communicating with your students is both exciting and rewarding. The more you know about the way in which students perceive themselves as learners, about their strengths and weaknesses, and about how they process information, the better you can modify and adjust your teaching to nurture the learning process.

5

Writing and Cognition: Implications for Mathematics Instruction

Patricia Ehrich

> If I commit these words to paper it is because only so shall I know my thoughts; only so may I calmly reflect on it; and so at length win from this bewilderment some glean of understanding.
>
> —Lindsay Clarke,
> *The Chymical Wedding*

During the past decade, the recommendations put forth by NCTM and others suggest the need for new goals in the teaching of mathematics. Rather than view mathematics teachers as transmitters of knowledge, educators are being asked to view themselves as facilitators in a classroom where students construct their own mathematical knowledge. To promote this type of interactive learning, teachers must begin to look for a variety of learning tools that allow students to engage actively in the construction process. Students must have access to methods that promote cognitive development, that is, methods that help students activate prior knowledge, combine this knowledge according to different models, and form new knowledge structures. These learning tools must give students and teachers the means to communicate ideas from student to teacher, from teacher to student, and from student to self. One such tool for mathematics educators is students' writing about mathematics.

As teachers of mathematics, we need to afford students the opportunity to build their own understandings of concepts and processes. I do my best teaching when I am able to design questions that lead students to build a deeper understanding of mathematics. Good teaching requires the ability to ask compelling mathematical questions that provide opportunities for students to build, tear down, rebuild, reflect on, and eventually own what they have constructed.

WRITING AND BUILDING MATHEMATICAL UNDERSTANDING

Writing is one of the ways in which teachers of mathematics can give their students these building and reflecting opportunities. Four years ago I began writing with my middle school mathematics students to give them a vehicle for thinking about their mathematics. Although my students use journals for a variety of types of writing, this paper focuses on process writing, or writing that promotes students' thinking about the process they are going through to solve a problem. Process writing is an invaluable

tool that allows students to become aware of their own thinking and reveals the cognitive aspects of the problem.

The notion that the act of writing directly influences the writer's thinking process is well documented (Emig 1977; Britton 1982; Applebee and Langer 1987). Journal writing allows students to use their own language and serves as a nonthreatening arena for expressing their thoughts personally and sometimes nonnumerically. But more important, I have come to see writing in mathematics class as a way to promote specific types of cognitive processing. Writing about mathematics helps students reflect on their own thinking, a process commonly referred to as "metacognition."

When students write in response to mathematical situations, their metacognitive awareness expands in several distinct ways. After studying several hundred of my students' journals, I identified several types of cognitive processing. The student writing samples presented in this essay demonstrate four distinct cognitive processes that can be stimulated through writing assignments: cognitive dissonance, affirmation, exploration, and the "aha" experience.

Writing and Cognitive Dissonance

Cognitive dissonance results when unsettling information comes to light while a student works on a mathematical problem. This information challenges or assaults what the learner knows and believes. When learners experience this conflict during their writing exercises, it increases the likelihood that learning will take place. Learners have an opportunity to shift their constructs or beliefs as they resolve the conflict.

Cognitive dissonance was particularly evident when I asked an average seventh-grade class to locate the number 0.2 on a number line as shown in figure 5.1 and explain why they located it where they did.

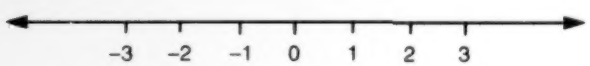

Fig. 5.1

The question engaged many students, clearly revealed what they thought about the decimal's relative size, and gave us good reference points for our initial discussion of decimal concepts. Many students used this writing as an opportunity to clarify for themselves their understanding of decimal numbers. Once they express themselves in writing, it is possible for students to begin to know what they believe about decimal numbers. For one student, the dissonance began before he wrote. He sat for several minutes after the question was posed and stared down at the number line he had drawn in his journal. I went over to his desk and suggested that he put the number on the line where he thought it should go even if he was not exactly sure of its location. "I will," he said. "I'm just thinking." The writing exercise gave this student an opportunity to ponder and reflect on what was for him a compelling question. Writing assignments, whether or not they involve dissonance, make students think!

Many students were greatly confused about where 0.2 should go on the number line. The cognitive dissonance experienced by many of my students is demonstrated in the entry in figure 5.2. Their faces showed discomfort and frustration as they attempted to respond to what I thought was a simple question for seventh graders who had worked with decimals and number lines for several years. I was surprised to find that half my seventh graders could not successfully locate 0.2 on the number line. Many of the incorrect answers that I saw as I circulated around the room looked like that in figure 5.2.

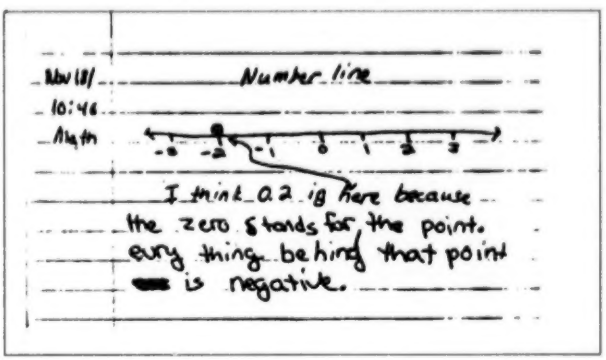

Fig. 5.2

The discussion that followed made clear that as many as half the students in the class experienced great difficulty with this problem. Writing a justification for where they located 0.2 made students aware that they were not sure of its actual value. They experienced the discomfort that comes with cognitive dissonance, which led them to wrestle with their beliefs. After this writing exercise and the discussion that followed, many students were

convinced that 0.2 was not located at –2, but a few were clearly not convinced. One student wrote this in a follow-up entry the next day:

> I know this is the wrong answer, but I really should have gotten the answer right. Because everything behind the decimal should be a negative number. That is my reason why it makes sense.

Although on one level this student accepted the class's argument that 0.2 was located between 0 and 1, he did not believe this to be true and did not demonstrate the level of ownership of this concept that some of his classmates had experienced through this exercise. More writing and discussion was required to help this student achieve an understanding of the concept of decimal numbers. But as a result of his struggle with cognitive dissonance, he now owns the concept.

The example in figure 5.3 shows cognitive dissonance created when a sixth grader confronted the rule for divisibility of any number by 4. The writing assignment was to explain why any number whose last two digits are multiples of 4 is evenly divisible by 4. The dissonance experienced by the student is shown by his confused attempts to state the rule. He says "two digits that add up to a multiple" in an apparent reference to the rule for divisibility by 3 or 9. This student has articulated what he perceives to be a contradiction between rules. His writing gave voice to this dissonance and gave me an opportunity to discuss with him the actual workings of each of these rules for divisibility. My response to his dissonance was to begin a written dialogue with him on the issues and questions he raised regarding the divisibility rule. Since my students write only on the right side of their journals, the left side is reserved for my comments and questions and their response to my response. Although the problem may not be resolved instantly for the learner, this method of dialogue gives each student an opportunity to construct an understanding of a problem or concept in his or her own terms and possibly resolve the cognitive dissonance that mathematical problems create.

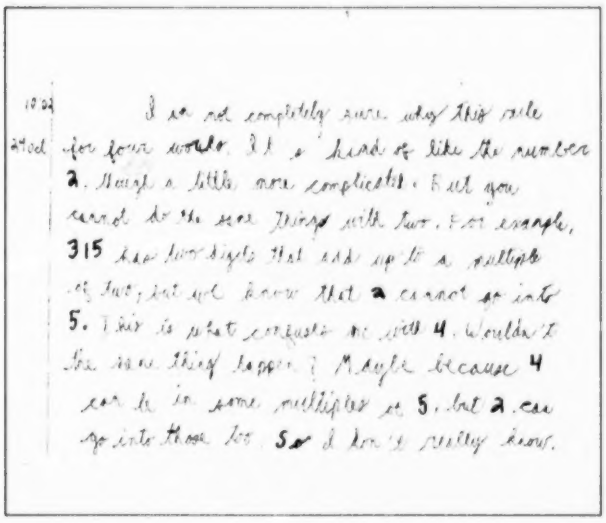
I'm not completely sure why this rule
for four works. It's kinda like the number
2. Kinda a little more complicated. But you
cannot do the same things with two. For example,
315 has two digits that add up to a multiple
of two, but we know that 2 cannot go into
5. This is what confuses me with 4. Wouldn't
the same thing happen? Maybe because 4
can be in some multiples of 5, but 2 can
go into those too. So I don't really know.

Fig. 5.3

Writing and Affirmation

When I talk to teachers about writing in mathematics class, someone will occasionally ask why students should write about problems for which they already have answers. My own research (Ehrich 1991) shows that writing allows students to affirm and strengthen their understanding of a particular piece of mathematics even after they arrive at a solution. They confirm their ideas and beliefs about a particular mathematical skill or concept and thus evaluate the accuracy of their own thoughts.

In the entry in figure 5.4, another student is asked to place 0.2 on a number line and explain why the location was selected. Although she is somewhat tentative about the number of intervals between 0 and 1 in centimeters (as evidenced by the "I think"), the rest of her entry shows a clear understanding of the correct location justified with an appropriate explanation.

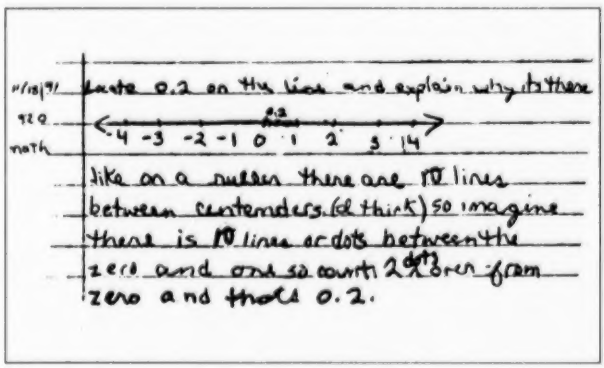
Locate 0.2 on the line and explain why it's there
0.2
-4 -3 -2 -1 0 1 2 3 4
like on a ruler there are 10 lines
between centenders (I think) so imagine
there is 10 lines or dots between the
zero and one so count 2 dots over from
zero and that's 0.2.

Fig. 5.4

Two days later, after considerable discussion about decimals, students were invited to revisit this question, and this student wrote with confidence about her original entry (see fig. 5.5). Her language shows the benefits of class discussion, since she

now refers to the intervals as "units." But her original answer has been affirmed by writing and by the classwork that followed. This experience strengthened her belief in the concept of decimal size.

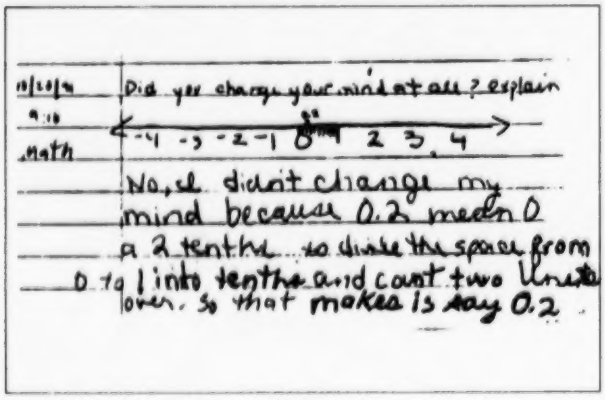
11/10/94
9:10
math
Did you change your mind at all? explain
-4 -3 -2 -1 0 1 2 3 4
No, I didn't change my mind because 0.2 means 0 a 2 tenths so to divide the space from 0 to 1 into tenths and count two units over. So that makes is say 0.2

Fig 5.5

Writing and Exploration

Writing gives students an opportunity to explore a problem that the directive "solve this problem" does not. Exploratory writing is done when learners are willing to be adventurous and leap into a problem even though they may not have a route to the solution in mind. These problem solvers allow their minds to wander freely, and writing can facilitate this process.

The journal entry in figure 5.6 was written by a seventh grader who was asked to solve the equation $3x + 9 = x + 6$ and to explain each step in her solution or to note where she experienced difficulty. This student had never attempted to solve an equation that contains variables on both sides of the equals sign, as she notes in her last paragraph. She is, however, a risk taker and begins to explore the possible methods for solution on the basis of her experience with simpler equations.

We also see evidence in this entry of cognitive dissonance. The student tries to "divide off" by 3 and is not confident of her answer. Before her second attempt, she demonstrates metacognitive thinking as she reflects on why she is having difficulty with the problem and realizes that the problem is expressed in a form different from that to which she is accustomed. Her entry demonstrates the value of allowing students to explore new problems and draw on their existing mathematics to create new levels of understanding for themselves.

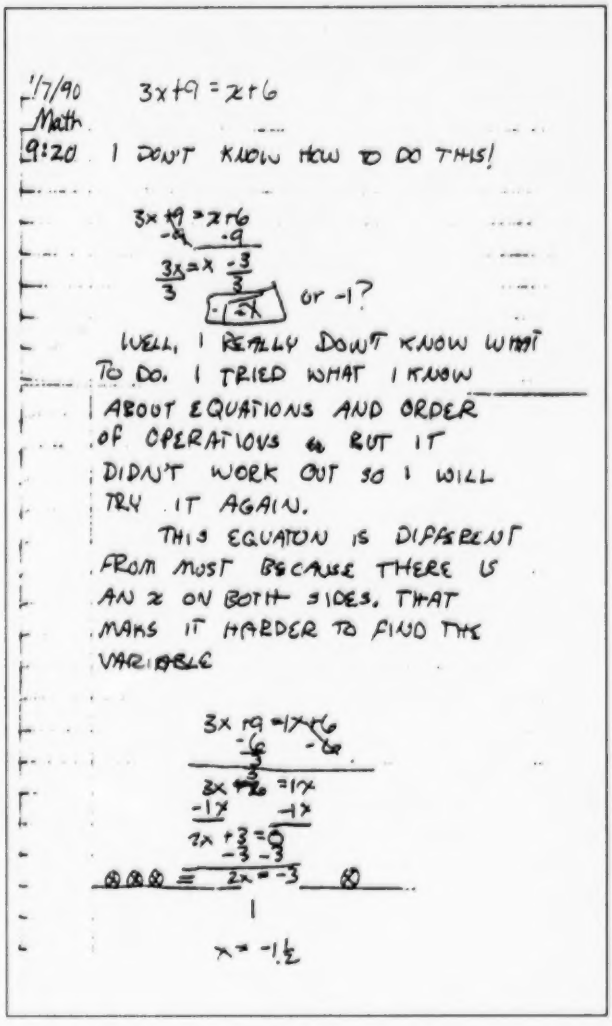
1/7/90
Math
9:20
3x+9=x+6
I DONT KNOW HOW TO DO THIS!
3x+9=x+6
-x -x -9
3x/3 = x-3/3
x = -3
or -1?
WELL, I REALLY DONT KNOW WHAT TO DO. I TRIED WHAT I KNOW ABOUT EQUATIONS AND ORDER OF OPERATIONS & BUT IT DIDNT WORK OUT SO I WILL TRY IT AGAIN.
THIS EQUATION IS DIFFERENT FROM MOST BECAUSE THERE IS AN x ON BOTH SIDES. THAT MAKS IT HARDER TO FIND THE VARIABLE
3x+9=1x+6
-6 -6
3x+3=1x
-1x -1x
2x+3=0
-3 -3
2x=-3
x=-1½

Fig. 5.6

In the entry in figure 5.7, an eighth grader was asked to use a two-column format to "simplify the expression $72 - 3(5 + 8)$ in as many ways as you know how and explain each step of your simplification." She closed her entry by stating that these were the only two ways that she knew of simplifying the expression. Her willingness to explore multiple methods led to greater flexibility and understanding.

"Aha" Writing

Students' exploration of problems sometimes produces what researchers call "aha" writing. This writing exhibits spontaneous discovery by the learner during the act of writing. Although not as common in students' journal entries as cognitive dissonance

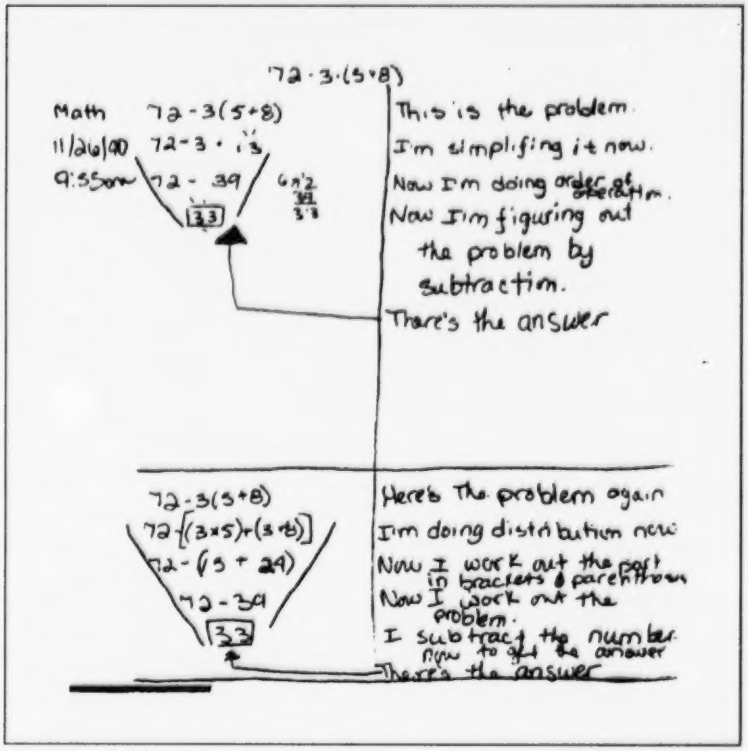
72 · 3 · (5 + 8)
Math
11/26/90
9:55am
72 - 3(5+8)
72 - 3 · 13
72 - 39
33
This is the problem.
I'm simplifing it now.
Now I'm doing order of operation.
Now I'm figuring out the problem by subtraction.
There's the answer
72 - 3(5+8)
72 - [(3×5)+(3×8)]
72 - (15 + 24)
72 - 39
33
Here's the problem again
I'm doing distribution now
Now I work out the part in brackets & parenthesis
Now I work out the problem.
I subtract the number now to get the answer
There's the answer

Fig. 5.7

or exploration, it may be one of the most important reasons to write in mathematics class. According to writing theorists such as Emig and Britton, writing is a unique tool for fostering original insight and developing new knowledge (Emig 1977; Britton 1982). Similarly, "aha" writing in the mathematics classroom allows ideas to appear spontaneously.

The student journal entries in figures 5.8 and 5.9 demonstrate elements of the "aha" experience. In the first example, the student is attempting to "define the term *consecutive integer* in your own words." She claims to have no idea of its meaning, but then proceeds to list consecutive integers correctly. As I watched the student write in class, it was clear to me that she had made a sudden discovery during her writing. Her question mark with an exclamation point suggests a sudden change when she went from being unsure about her answer to being certain that she was right.

In the second example, the student gives an initial response, "I do not agree," but in the course of working on the problem suddenly finds that he does in fact agree with the statement. He was responding to the following: "Do you agree or disagree that there are many ways to solve the following equation? $2x + 8 = x + 11$. Give evidence to support your position."

In both instances, these students made a spontaneous discovery. It is difficult to determine what sparked their insight, but the process of writing seems to have been an important factor, which justifies incorporating regular writing experiences in school mathematics programs.

HOW TO GET STARTED

Some formal structure is helpful in a mathematics writing program. At my middle school, a team of seventh- and eighth-grade teachers began an interdisciplinary journal-writing project five years ago. Students carry a single composition notebook to all four of their core classes—English, science, social studies, and mathematics—where they are occasionally asked to write journal entries

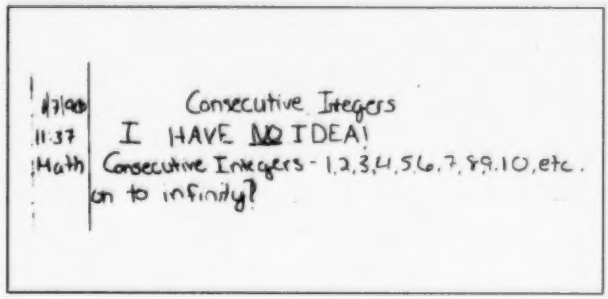

Fig. 5.8

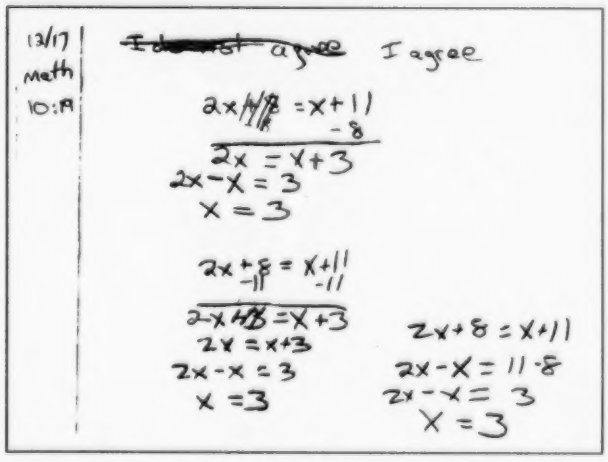

Fig. 5.9

such as the ones shown above. Although the students are given guidelines for journal entries, they are encouraged to write freely and honestly about the questions and problems they are given.

To ensure that the journals are risk-taking vehicles, the students are told that their entries will not be judged right or wrong. Only the teachers on the team see and comment on the entries, and their comments support and encourage the students. Through the writing experiences and the subsequent discussions or the written dialogues on their entries, many students demonstrate increased awareness of the cognitive processes involved in mathematics and develop deeper understandings of concepts, terms, and procedures.

Although we do not grade journal responses for correctness, the students are given a score from 1 to 10 each marking period for their effort, consistency, and honesty in responding. Although the students sometimes express dismay and reluctance when asked to write in mathematics class, I have found that they appreciate a nonthreatening format through which they can express mathematical ideas.

CALLS FOR REFORM IN MATHEMATICS EDUCATION

Reshaping School Mathematics (National Research Council, Mathematical Sciences Education Board 1990) and the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) contain dramatic directives for restructuring school mathematics on all levels. Both documents see the development of language and communication as integral parts of the restructuring of our mathematics programs. These two reports indicate clearly the need for more meaningful activities that engage students in active thinking and learning. Writing is an appropriate activity for mathematics class because it promotes and supports cognitive processing by providing opportunities for diverse thinking. Writing belongs in the mathematics classroom and should play a part in the redesign of mathematics curricula.

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6

Conflicting Goals in the Mathematics Classroom

Ann Oaks

Each term a number of my first-year college students—often up to a third of the class—find themselves unable to keep up with the requirements of our calculus course despite the fact that each of them seems to have an adequate mathematics background and a few have even completed high school calculus. While their classmates progress through the topics in a satisfactory manner, these students become hopelessly mired in the day-to-day mechanics of the course. A number of them fail.

I have discovered that many times these students and I enter the classroom with opposing goals. I want to guide them through the ideas and logic behind the topics we cover, to help them understand basic questions that give rise to mathematical theory, and to encourage them to think about mathematics and to use intuition in their attempts at problem solving. They want to memorize algorithms.

As a result of our conflicting goals, we often are working at cross-purposes. While I am attempting to broaden their skills in problem solving, they view problems for which they have no algorithms as unsolvable and they are unable to engage in finding a solution. Whenever I present a theory, they look for an algorithm in it. Derivations are memorized, ideas are ignored, and proofs are stripped of meaning.

Over a decade ago, when I began teaching at the college level, I was puzzled by my students' reactions to mathematics. Although I realized that we were many times at odds with one another in the classroom, I did not know why they resisted my efforts to engage them in problem solving and why they seemed incapable of following any discussion of theory. Furthermore, I did not understand how these academically bright students could fail at entry-level mathematics. In an effort to uncover factors influencing my students' behavior, I began a research project aimed at describing their world of mathematics from both affective and cognitive viewpoints.

My interests were focused on two areas—the difficulty my students encountered in solving problems and their inability to deal with mathematical theory. The literature on conception of knowledge or belief systems offers a framework for studying both these concerns. Alan Schoenfeld, in his book *Mathematical Problem Solving* (1985, p. 15), defined *belief systems* as "one's 'mathematical world view,' the set of (not necessarily conscious) determinants of an individual's behavior." He described belief as dictating the focus and direction of a problem-solving attempt. For example, because one pair of students in his study did not view the proof as important to the context of the problem, they failed to use the results of a proof they had just written that would have led to the solution of the problem on which they were working.

In my study (Oaks 1987), I characterized conception of mathematics as the view students hold of mathematics, how they would describe mathematics, and what they believe is required in learning and doing mathematics. Although Alan Schoenfeld's definition of belief systems is very similar, I studied conception in the general context of the mathematics classroom. This context includes problem solving but also involves looking at other aspects of student experiences with mathematics, such as discussion of theory in the classroom, notetaking, studying, and taking examinations. Although problem solving is a large part of our students' attempts to learn mathematics, the way in which they interpret and use the theory and ideas of mathematics is of equal importance, as is the view they construct of mathematics itself.

The conception students have of mathematics determines their perception of their experiences in the classroom—the way in which they interpret classroom activity. Conception also dictates their goals and expectations for their quest to learn and do mathematics, which in turn determine their behavior. For example, students who view mathematics as a subject demanding recall of arbitrary, fixed instructions presented by the teacher will enter the classroom with the expectation that the teacher will present such instructions and with the goal of recording the instructions. Therefore, their perception of mathematical activity will focus on algorithms and they will interpret classroom activities in a rote way.

Because students are often unable to describe their conception of mathematics directly, I used a variety of data-gathering methods to record their mathematical behavior and their descriptions of their experiences with, and opinions about, mathematics. I used the data I collected to construct a description of my students' conception of mathematics.

In order to describe their behavior in a problem-solving setting, I videotaped pairs of students as they attempted solutions to a list of algebra problems. These tapes were transcribed and studied in an attempt to uncover the strategies and techniques they employed while solving problems. In addition, I collected weekly journals from students over the course of six terms so that I would have the opportunity to study how they expressed their beliefs about mathematics outside the constraints of a problem-solving situation. These journals also served as a means by which I could get at my students' affective responses to mathematics by listening to their feelings about themselves as learners of mathematics. In order to measure students' attitudes and conceptions of mathematics on a larger scale, I designed a survey that was completed by students at five colleges and universities in central New York State.

After these data were collected and analyzed, I conducted a series of in-depth interviews with four students, Roland, Katie, Patricia, and Andrew. The questions for these interviews were generated by the results of the survey, the journals, and the videotaped problem-solving sessions. My goal was to describe students' conceptions of mathematics, students' affective reactions to mathematics, and the interaction of these two factors in determining the results the students experience in the classroom. (For a compete description of this study, see Oaks [1987].)

UNDERSTANDING STUDENTS' CONCEPTIONS OF MATHEMATICS

When I began listening to students who have difficulty with mathematics and observing their mathematical behavior, I confirmed my initial hypothesis that they hold a rote conception of mathematics restricted to meaningless symbol manipulation. They see mathematics as a system of types of problems, each of which comes with a prepackaged solution or algorithm that is to be followed exactly as the teacher presents it.

Clearly, most experienced learners of mathematics approach a new topic by initially focusing on procedures handed down by previous generations of mathematicians. We certainly cannot generate each and every formula and algorithm on our own. However, in this process most of us are aware that these procedures come from human beings who did generate their own work—who, when searching for answers to meaningful mathematical questions, developed the definitions, theorems, and algorithms that make up the topics we encounter in the classroom. Furthermore, we understand that in order to take ownership of the mathematics we are studying, we must attempt to put ourselves in the shoes of the mathematicians who developed the theory so that we will understand the inspiration and reasoning behind the algorithms we learn.

Three of the students I interviewed are completely unaware of this domain. Mathematics for them is not owned or generated by anyone. It is just "out there," as one student explained, consisting of fixed and arbitrary procedures that must be followed precisely as they are written. This mathematical world does not allow for reason or logic, and its restricted nature bars individual intuition and creativity. Students are therefore limited to learning mathematics by memorization.

Students like Katie, Roland, and Patricia not only hold this rote conception but they do not consider that others may view mathematics differently. They do not even recognize that others value the system of logic and reasoning that forms the foundation of mathematics. Instead, they believe that mathematics for everyone is restricted to the process of recording and memorizing rote procedures. Therefore, their perception of mathematics and the activities connected to it is formed by the only structure that makes sense to them—algorithms.

Even an axiom as basic as the distributive property is interpreted by these students as a step in an algorithm. Several of my students explained that this property tells them what to do when faced with an expression of the form $a(b + c)$. They believe that they must multiply a and b and add the result to the product of a and c. Otherwise, the "answer" will be wrong. Consider Katie's explanation of how she would teach the distributive property to an elementary school student. We used the example $2(3 + 4) = 2(3) + 2(4)$ for our discussion:

Katie: So take whatever is outside the parentheses and multiply it with whatever is within the parentheses. (*She writes* $2(3) + 2(4) = 6 + 8 = 14$.)

Ann: What if the student said to you, "I'd rather do that by adding 3 to 4"?

Katie: Well, it's not the right order of operations. ... If you do that in a different order, you won't come out with the correct answer.

Ann: So it's likely that if I add 3 to 4, I might not come out with the same answer?

Katie: You won't. Well ... (*as she does the arithmetic in her head*) you will.

Ann: You will with this one.

Katie: OK. But then if the student notices that, you pick an equation where it's not true.

Ann: So what you're saying here is that when you see $2(3 + 4)$, you have to say $2(3)$ plus $2(4)$? That's what the law tells you?

Katie: Right.

Katie left the interview believing that she might get the wrong results if she attempted to add b to c before she multiplied by a. Clearly, she was reading the equals sign of the distributive property as true only from left to right, not vice versa. She described this property as "a tool to set up an equation so you can solve it," in other words, a step in an algorithm.

The students in my study also explained that mathematical terms such as *axiom*, *theorem*, *proof*, and *formula* all mean "method of solution," the only interpretation that makes sense to them in the mathematics domain. They use *formula* most often in this context. However, Patricia defined a "proof" as an algorithm written in words rather than symbols. When she mentioned that proofs are sometimes helpful, I asked for an example. Here is her reply:

Patricia: I think you did it over here—this type of thing where you put it in English.

Ann: OK. (*Reciting the solution to an equation that had been written previously*) "Square both sides. Add 16 to both sides. Take the square root of both sides."

Patricia: Right. Write it in English instead of numbers.

Roland speculated that since he had seen the word *theorem* used at the top of boxes in his textbook, a theorem is likely to be an important formula (method of solution). Katie explained that *theorem* simply means method of solution.

The meaning these students attach to *understanding* is also generated by their conception of mathematics. Because they cannot talk about understanding in mathematics other than to outline the steps necessary to solve a particular problem, they define understanding as the ability to retrieve a correct algorithm from memory. Consequently, when they explain that understanding is important to the process of learning mathematics, they are simply saying that knowing the algorithm for a particular problem is a serious requirement of the discipline.

For example, many of the students who completed my survey explained that memorization is necessary in learning mathematics but understanding the

concept is more important. A nineteen-year-old student wrote the following:

> Memorization shouldn't be the most important. Learning and understanding the math is far more crucial. Once an understanding is gained, the application follows easily.

Such a response indicates that this student has internalized the fact that understanding is necessary in order to succeed at mathematics. However, when asked how one might know whether mathematical understanding has been achieved, her response dealt solely with the process of applying an algorithm:

> If I can do the homework and explain to myself the procedure, then I feel I understand the material.

She also described as "very true" the statement that "to solve math problems you have to be taught the right procedure or you can't do anything," indicating that she holds a conception of mathematics that is focused on rote recall of algorithms.

When presented with a problem to solve, students with such restricted conceptions of mathematics engage in a three-step process. They first attempt to recognize or categorize its type, they then attempt to retrieve the appropriate algorithm from memory, and finally they follow the algorithm to produce an answer. If successful in this procedure, they claim to "understand the problem." If they fail at any step, they have no recourse but to give up and wait for someone else to solve the problem for them.

STUDENTS' PERCEPTION OF CLASSROOM ACTIVITY

The results of my study indicate that students' conception of mathematics dictates their perception of mathematical activity, that is, how they create meaning in a mathematical setting. Because they see and hear only algorithms, the meaning they take from my classroom is constructed from this focus. They expect that I will introduce examples and solutions in a clear, organized manner and that they, in turn, will follow each of these algorithms until they feel comfortable with the procedure and can remember all its steps. They see homework exercises as vehicles for practicing instructions presented in class so that they will be able to "solve" problems on examinations. Consequently, they expect that the primary business of the classroom will be the presentation of different types of problems and that algorithms will be provided for each problem they encounter.

Therefore, my students and I enter the classroom with conflicting goals and expectations, causing us to work at cross-purposes. Whereas I may attempt to explain a concept and talk about theory or ideas, my students are looking for procedures. As a result, they turn mathematical arguments into algorithms, and because memorization is the only way in which these students can deal with mathematical material, they memorize required proofs and derivations line by line. Moreover, they ignore any activity that cannot be interpreted within their algorithmic framework.

A rote conception of mathematics also dictates students' behavior in a mathematical setting. Students like those in my study attempt to learn mathematics by recording an algorithm and practicing it on appropriate problems until they have memorized each of the steps in its procedure. Roland described the process by which he studied mathematics:

> First you have to memorize the formulas if you are ever going to use them again. And they have to be in your head, right at hand. And then you have to practice using those. In a lot of things like economics and political science, you study what you have to study, it's there. But with math you have to know the formulas and then you have to practice them.

Katie explained that when she studies mathematics, she finds an algorithm and does a few problems:

> Practice makes perfect. Get a theorem, go through a few examples and then go into the book and find as many different ways to do it and use it and apply it as you can.

Notice that Roland used the word *formula* in place of *method of solution;* Katie used *theorem.*

HOW CONCEPTION CONFLICTS WITH CLASSROOM GOALS

How does this rote conception of mathematics keep students from becoming good problem solvers and from recognizing the ideas and theory of mathematics? If they are working hard at memorizing algorithms, why do they fail to meet the demands of our mathematics courses?

Consider the difficulty they encounter in solving problems. In his classic book, *How to Solve It*, George Polya (1973, pp. xvi–xvii) outlined a general problem-solving process used by most mathematicians. He suggests that a good problem solver will—

- first, understand the problem;
- second, devise a plan;
- third, carry out the plan;
- fourth, look back.

The rote conception of mathematics held by students like Katie and Roland makes them unable to engage in problem-solving activity as described by Polya. An analysis of the videotaped sessions suggests that students who define "understanding" as the ability to recall a correct algorithm from memory interpret Polya's outline in terms of their own three-step process mentioned above: identify the type of problem, recall the correct algorithm, and apply that algorithm to arrive at a solution. Because they identify most problems mainly on the basis of any pattern of symbols contained in the problem, they seldom even read the question. Furthermore, they interpret Polya's instructions to "devise a plan" as "pull the necessary algorithm from memory." They then "carry out the plan," although "plan" for these students simply means "algorithm." Moreover, unless they are told to "check," they ignore Polya's last step altogether, as did every pair of students who participated in the videotaped sessions in my study.

Clearly, my students' approach to problems bars any attempt to answer questions for which they have no algorithm. The activity of generating their own method of solution simply does not fit within their conception of mathematics. The following incident that occurred during a precalculus class is a good example of how students restricted by a rote conception of mathematics respond to nontraditional problems. I asked my class to show that three points on the Cartesian plane were collinear. Immediately, Chris, one of my students, explained that he did not know "the formula for collinear." When I replied that there is no such formula, he responded, "How do you expect us to solve a problem if we haven't been shown how?" Several members of the class nodded their agreement. They were clearly puzzled about why I would expect them to solve a problem without first showing them how. Of course, at the time I was baffled by their refusal to generate a solution on their own when I knew they had the resources that would allow them to do so.

Now that I have a better understanding of my students' conceptions about mathematics, I interpret this classroom exchange differently from the way I did prior to my study. First of all, the difficulty my students and I had in communicating was caused partially by the meaning we attached to the word *formula*. I thought Chris was using this term to mean an equation such as the quadratic formula or the distance formula; instead he was telling me that he did not have an algorithm or a list of instructions for solving the problem. But overriding the entire experience was the difference in my expectations and those of Chris and his classmates. I wanted them to develop their problem-solving skills by generating a solution to a question they had not seen before. Because of their mathematical beliefs, my students were unable to do so. They were not lazy or unprepared, but I was asking them to do something that did not fit within their conception of what mathematics is all about. Consequently, they could not even attempt a response to my question; furthermore, they did not understand why I might expect them to do so.

Even traditional questions become difficult for students who rely on rote recall of algorithms in doing mathematics. Their method of studying and their three-step process for doing problems may work well on homework assignments that contain one or two types of traditional questions. However, it generally fails on examinations covering a variety of topics. Patricia explained that she has trouble identifying the correct algorithm for the problem at hand:

> I could remember the formulas [algorithms], but I would be remembering two or three formulas at a time and trying to figure out which one fit the equation. There are so many different equations and you have to know how to go through each one. Some equations are done for different things, and I never knew which one I was looking at.

Roland described final examinations as "nightmares" because he could not remember all the methods of solution studied during the term:

> Comprehensive finals in math are my worst nightmare because I have to go back and try and study everything. I'd have to cover the course again because ... that for some reason doesn't hold in my mind.

Katie explained that if she gets stuck on a homework exercise, she can look at her notes or the examples in her textbook. However, on an examination when she cannot look up the correct method of solution, she is often uncertain about her choice of algorithm, erases her work, and starts with a new method she thinks might be more applicable. Sometimes this approach degenerates into a process of trying out various algorithms that leaves her paper a scrambled mass of symbols.

Also, because students like Patricia and Katie identify an algorithm solely on the basis of the pattern of symbols in a problem, they are likely to recall the wrong set of instructions for a question. When I ask my students, for example, to find the domain of a quadratic function such as $y = x^2 - x - 6$, I can be certain that a number of them will simply locate the roots of the polynomial. The pattern of symbols in the expression activates the "find the roots of a quadratic" algorithm, and students answer an entirely different question from the one being asked. Consequently, they have difficulty passing tests.

Furthermore, such students have no way of improving their problem-solving performance. When they experience poor grades, they initially spend more time studying. However, increased effort takes the form of memorizing and practicing algorithms, a process not likely to increase their ability to identify the correct method of solution for the problem at hand. As a result, grades appear independent of effort, and failing grades many times follow long hours of work. They become trapped in a cycle of working hard and failing, a frustrating situation that encourages them to escape from mathematics rather than to work at it.

In addition, students who view mathematics as consisting of nothing more than long lists of rote problems and algorithms are unable to understand the logic, reasoning, and human exploration behind mathematical theory. They cannot understand a discussion of theory, nor can they marvel over a wonderful idea or enjoy an elegant proof. Therefore, they see none of the beauty of mathematics. Moreover, because they are unable to construct mathematics in a meaningful way, they are unable to internalize mathematics as a body of knowledge. Mathematics is both threatening and boring for these students, and they are likely to reject mathematics because they are unable to take ownership of it.

WHAT CAN BE DONE?

A first step in our quest to help students achieve in the classroom is to understand their conceptions of mathematics and recognize that these beliefs determine their perception of what goes on in the classroom and their goals and expectations as they attempt to learn. Because students perceive mathematics through the filter of their conception, we must work to expand their mathematical world and to help them internalize the broader messages of mathematics.

A deeper understanding of mathematical concepts will help these students recognize that they can and should think about the questions and theorems they encounter in mathematics just as they ponder questions and theory in English and history. With this realization will come the ability to use reasoning skills and intuition along with memory, giving students a variety of methods by which to learn and do mathematics. More important, with a better grasp of the logic and ideas that make up mathematical theory, students are likely to find learning mathematics enjoyable.

But how do we accomplish this with students like Roland and Katie who come to us ready to record and memorize every word we speak and each symbol we write? How do we help them broaden their beliefs and encourage them to become participants in the mathematical process? How do we get them to recognize that ideas exist behind the procedures they work so hard to memorize?

If we are to change the way in which our students approach mathematics, to ensure that they are not simply reproducing material their teacher has given them, we must first make certain that they are in an environment that does not allow them to fall back into the familiar process of recording and memorizing. This task is not easy. If students are to begin thinking on their own and generating their own work, the center of activity must shift from the teacher to the students. The traditional process of "teacher telling, student listening" has likely helped create the rote conceptions many of our students hold and has forced them to be passive observers in the classroom.

When lectures are necessary, they should focus on concepts rather than procedures. However, since students with an algorithmic view of mathematics interpret even the most theoretical presentation in a rote way, whenever possible, students themselves should be active participants in the learning

process. They must do exercises that require reasoning, inquiry, and exploration rather than implementation of an algorithm. These activities can take the form of individual or group work in and out of the classroom, problem-solving sessions with or without the aid of a computer, or field projects. I engage my students in a variety of these activities.

Whatever the activity, students should be put into the mathematician's shoes so that they will come to understand the context of the topic they are studying as well as the logic behind their calculations. They should be encouraged to explore a question and develop and test their own procedures for resolving it just as a mathematician would.

I have found that asking questions that are different from those typically encountered in a mathematics course helps prevent students from responding in a rote way (Oaks 1991). Patricia explained that to her, *solve* simply means doing whatever procedure is being taught that week. Instead, if faced with instructions to "explain," "justify," or "explore," students have no alternative other than to stop rote activity and ask themselves, "What am I supposed to do here?" And although they may initially look to their teacher to be "shown how," in following directions they are encouraged to think about mathematics in a different way.

Whether students work alone or in a group, on a computer or with paper and pencil, questions of this type require that they write about their own thoughts and reasoning rather than simply repeat, word for word and symbol for symbol, what others have given them. Although difficult at first, these activities encourage students to stand back from their old approach to mathematics, examine the material they are working to learn, and reevaluate their approach to learning it. This is their first step in realizing that mathematics can and should come from within themselves as well as from the textbook.

David, a student in one of my calculus classes, was forced to look at his approach to doing mathematics when he encountered a question that asked for explanation. At the beginning of the term, he wrote the following explanation of why he was unable to solve a nontraditional problem:

> This question is hard. I have been thinking about it off and on, but I still do not know where to begin. I do not think that people have thought about this kind of problem except for mathematicians. I have yet to come across this problem. I have always just memorized problems and then answered them on a test or quiz. You rarely ever think about what you are memorizing.

When he explained that he had always "memorized problems" and that he rarely thought about what he was doing, he was forced to consider that other ways of doing mathematics might be possible. This realization and the fact that his old approach was not working prompted him to reevaluate both the material he was attempting to learn and the techniques he employed in learning it. Course-evaluation comments such as "I never looked at math this way before" and "She wants us to understand 'why' as well as 'how'" suggest that my students are seeing mathematics in a new and more meaningful light.

At the beginning of a term, students are likely to resist my efforts to get them to change their approach to doing mathematics. They do not know how to respond to these new questions, and they resist writing and group work. All these activities are new, and in combination, they require that students give up their old, comfortable (although not necessarily successful) ways of learning. Students are also uncertain that these new methods will work for them.

However, as their writing and communications skills improve, students get used to these new expectations. They begin to realize that they must work for conceptual understanding in their study of mathematics, and most begin to take ownership of this new approach to learning. After their initial experiences with these new activities, my students begin to form their own informal working groups outside class, and they begin to resort to writing about mathematics even when they are not specifically required to do so. Writing becomes a new form of communication for them. In addition, they begin to talk about mathematics in a different way. Instead of asking, "How do you do this?" they are more likely to make observations about the ideas we have discussed in class.

CONCLUSION

Activities that are focused on ideas as well as procedures can open the door to a new conception of mathematics that allows students to internalize and take ownership of the mathematics they are studying. When students' conceptions about mathematics

expand to include an understanding of the ideas, their expectations and goals in the classroom will change. They will understand that learning mathematics involves more than simply rote procedures and algorithms; they will begin to see mathematics as a discipline generated by human thought, discovery, and invention.

However, we must understand that we cannot change our students' beliefs about mathematics in just one course. Rather, through continued experience with exploration, questioning, conjecture, and experimentation, our students will evolve into good problem solvers who appreciate the beauty of mathematics. Ideally, this process should begin in elementary school and be continued, reinforced, and expected by each of us.

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Broadening Students' Conceptions of Mathematics: Theory into Action

Dorothy Buerk

Lee, who wrote the statement that opens the introductory chapter of this collection of essays, experienced a change in his views about mathematics early in the semester he spent in my class as he began to recognize and listen to his own ideas and gain satisfaction from expressing them. He wrote an assigned paper explaining to a sixth grader how and why we multiply fractions in the way we do. He reflected afterward on that experience:

> We worked in small groups first and I extended this group activity to my individual work. Surprisingly, the dreaded "identifying and comparing process" did not appear in my figuring of the problem. I had no example to follow. I could not merely solve the problem using a previously specified method.
>
> I like writing mainly because I receive satisfaction when I finish a solidly written piece. While writing I am, at the same time, thinking and understanding what I'm writing about. Until the fractions paper I could not say the same about doing a math problem. I thought out and understood what I was doing while I was figuring how and why we multiply fractions the way we do. I felt like I was writing my own chapter of a math textbook.
>
> My doing the fractions problem well produced a satisfaction that sparked my interest in math. Furthermore, because I did the problem without referring to any examples of any supposedly proper methodology, I controlled the problem. I chose to take the problem in the direction I did. Finally, the ability to be creative made me want to do the problem and do it well.

Lee was surprised that he did not have an example to follow, but since he did not, he "thought out and understood" through his writing as he did when writing in other subjects. Without an example to dictate his direction, he felt in control to choose his own plan of action. For Lee, being creative meant having some personal input into the mathematics that he studied, which helped him become actively engaged in his own learning.

A DEVELOPMENTAL FRAMEWORK

I work mostly with college freshmen, many of whom would prefer not to be taking a mathematics course. Many begin my courses with the rote conception of mathematics described in this collection of essays. As I listen to my students and read their metaphors for mathematics written early and late in a semester, I begin to see indications that the first step toward a broader conception of mathematics is taken when a student begins to own his or her ideas, as Lee has described. This step seems to involve acknowledging that mathematics was, and is, made by people—people from different cultures; "famous" people; and ordinary people, including students. Once students grasp this notion, they can move on to feel ownership of the theory, the algorithms, or

the procedures that are so necessary in mathematics. As students broaden their conception of mathematics, they seem to pass through three distinct positions:

1. Seeing all mathematical knowledge as external, fundamental, and black or white
2. Acknowledging personal thoughts and ideas, often in a subjective way
3. Integrating known theories and new, carefully developed, individual ideas into an internal model and thereby gaining ownership of the mathematics they are learning

This process is similar to those described in two cognitive-development theories that I have found helpful. William G. Perry, Jr. (1970, 1981), using longitudinal data from a sample of college students who were predominately male, presents a sequence of positions describing intellectual and ethical development. Mary Belenky, Blythe Clinchy, Nancy Goldberger, and Jill Tarule (1986), working with women from diverse backgrounds, present various developmental positions that describe how these women view themselves and view knowledge. Although these theories speak to general cognitive development, I use their terminology and concepts to help me understand the way in which my students make sense of mathematics.

I find that many students enter my courses with the fundamental, black or white view, believing that they can have no ideas about mathematics and that every question has an answer and every problem a solution. Teachers and textbooks are the only sources of this knowledge. These students tend to learn mechanically, following the rules precisely. They believe that David Bock's Ten Myths of Mathematics are true. The metaphors for mathematics that Helen Gibson and I have gathered from high school students indicate that a large number of them also hold this view.

To shift to the second, more subjectivist position, students need an awareness of both diversity and uncertainty in mathematics and a more personal relationship to the subject. They need to accept their own ideas and experiment with them. In this shift they begin to value their own experiences with mathematical concepts but still focus on getting "the right answer." They can learn from the experience of their peers, for they come to realize that peers, as well as teachers and textbooks, have ideas worth considering. They begin to consider their own ideas but may not be able to communicate them. Writing opportunities and cooperative learning activities furnish excellent environments for students to consider a variety of mathematical approaches and place their own ideas in context.

By writing their metaphors for mathematics as Helen Gibson requests, by capturing their ideas in their journals as Patricia Ehrich suggests, or by annotating their assignments as Marcia Standera requires, students become aware of their own thoughts and the extent to which their teachers value this independent thinking. Students begin to realize that their ideas make sense, that mathematics can include their own ideas and those of their peers. As they provide explanations and justifications for Ann Oaks and verifications for Marcia Standera, they find that they understand the concepts more deeply. Through the cooperative learning experiences that David Bock describes, they become aware of different ways of seeing a mathematical question, different points of view on possible routes to solutions, and different interpretations of those results. They move away from the notion that their own thoughts don't belong in their mathematics classroom. They begin to share their own "thoughts in process," first in small groups or through their journal writing, then gradually with the whole class. They change their relationship to mathematics and broaden their view of what mathematics is and how they can learn it.

Students next become aware that their intuitions may deceive them and that some of their ideas don't work out. They share their ideas in process in their journals or in their small groups for feedback. They test their ideas, trying to find counterexamples or some convincing evidence to support their conjectures. Through this search, they come to value the process itself. They move away from a subjectivist view and learn to respect *both* established procedures *and* their own insights. They experience cognitive dissonance when a theory and their own insights are in conflict and come to know the pain and joy of real intellectual struggle.

In my courses I try to help students move through this developmental process of viewing mathematics. (See Buerk [1990] and Buerk and Szablewski [1993] for some specific examples of activities I use and students' responses to them.) As my students learn that they have mathematical ideas that make sense, they become empowered. They feel less at the mercy of

"magical methods" and become less fearful of the procedures and techniques needed to do mathematics. In addition, they become more relaxed. Once they feel free to think and to value their own thoughts, they become more receptive to suggestions and advice from others. They no longer ask what I want them to do but rather proceed in a manner that makes sense to them. They begin to remember more of the mathematics that they have learned in the past. They also begin to discuss the concepts with me and with their peers. The authors of all these essays have observed student growth of this type.

These authors believe that broadening students' conceptions of mathematics is necessary, but they know that the task will not always be easy. Many students fear or avoid mathematics, and although they may want to be more active, their passive behaviors are well ingrained and have served a purpose. The change in perspective described here and in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) will be dramatic for some students, and they may resist. Perry's (1978) cognitive-development theory has helped me here also. He stresses the importance of a mourning process, a need to grieve the loss of the old ways of thinking and knowing even as students advance. Students making this developmental transition sometimes appear to resist new ways of thinking when in fact they need support as they try to accept these new ways. We need to understand the dynamics of students' resistance as we work to empower our students.

PERSONAL REFLECTION AND PLAN OF ACTION

The teachers who wrote these essays have created learning environments that encourage active student involvement in their classes, have dealt with the resistance of some of their students, and have shared their stories with you. Now let's consider your own plan of action for empowering students in *your* classroom.

Begin with some personal reflection. You are a competent teacher! Reflect on your own teaching, on your own experiences with your students, and on your own practice as a teacher. (You may want to do this reflecting alone, or you may want to do it with a group of your colleagues.) First, list your strengths as a teacher and some successful lessons and projects that you have prepared for your students. Savor your responses! Praise for teachers seems to be on the endangered list these days.

Now to plan some course of action for your own classes, reflect on the following questions. Again think about whether you want to do this reflection with a group of colleagues or by yourself.

- What are my students thinking and believing about mathematics? If I don't know, how can I begin to find out?
- What do I believe, feel, and value about mathematics?
- Which of my beliefs, feelings, and values would I most like my students to embrace? Why are they important to me?
- How can I help my students see and experience mathematics in this new way?
- What resistance do I expect from my students? What can I do to learn to accept my students' resistance to change? What can I do to lessen this resistance? What can I do to prepare myself for my students' resistance to my new teaching strategies?
- What will hinder me from changing my students' experience with mathematics? (Maybe it is state or local curriculum mandates; the limited selection of textbooks; or the expectations of students, parents, administrators, school committee members, other faculty members, etc.) Which of these hindrances can I control? What can I do about them? Which of them are beyond my control? How will they impede my progress?
- How can I get started? What is most important to me to establish as a first step? What pace should I set? How much can I do and how quickly?
- What support system do I need to implement my plan? How can I create a support network for myself?

I hope that your reflection has helped you plan a journey in a new direction. Thinking about new directions reminds me of David Bock's story about his discussion with his colleague who had worked very hard to "lead and herd" his students carefully "around the end of this lake," always feeling rushed. After listening to David, his colleague realized that he needed to teach his students to swim instead. This is precisely what this volume is encouraging you to consider.

Breaking established patterns, questioning long-held beliefs, and examining familiar practices and views can be scary for both you and your students. Apprehension, even resistance, can be expected. The fact that you have read this far, however, suggests that you may be intrigued by something we have said. Continue to think about the issues raised in these essays. I hope this self-examination is the first step on what will become an exciting and rewarding journey for both you and your students.

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